Vault of Echoes

Volume I

Aaditya Paudel

A lore-infused puzzle codex for rigorous reasoning.
The Vault accepts only what survives a later reading.

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Vault of Echoes: Volume I

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This is a work of structured reasoning and mathematical imagination. All characters, settings, and narrative elements appearing in puzzle chapters are fictional and are products of the author's independent invention. Any resemblance to actual persons (living or deceased), institutions, or events is coincidental, or used solely as creative mythic framing.

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First Edition

Preface

Puzzles have been a part of human society for as long as civilization has existed.
I believe that we, as a species, have progressed because of our ability to find patterns—ranging from the most mundane to the extraordinarily complex.
This book is meant to gauge problem-solving ability in solvers of any kind.
Throughout history, puzzles have served as both a reflection of human cognition and a training ground for it. Today, as we approach the frontier of machine minds, they quietly take on a second role: a substrate through which intelligence—human or artificial—may sharpen and measure itself.
Artificial intelligence (AI) is, by extension, the intelligence created by humankind. In our pursuit of artificial general intelligence (AGI), we have—perhaps unconsciously—agreed that maximizing intelligence within specialized domains and merging those domains may be a way to create AGI. In this sense, we invoke the concept that "the sum of the parts equals the whole." One such titan among domains is mathematics.
Mathematics compresses structure into invariants: what must remain true when everything else is allowed to move. That is why the Vault leans on it so heavily.
The puzzles in this volume are designed to test logical reasoning, pattern recognition, and deep problem-solving ability across a range of domains, from combinatorics to geometry.
This is Vol. I of many Vault of Echoes volumes I plan to write, and I aim for the series to remain worthy of careful solvers for as long as it is read.
A note on characters: All the characters seen in this Vault are figments of my imagination. Starting with Neohm, the archivist portrayed after me, roles are defined meticulously so their consistency will remain stable across many upcoming volumes.
I hope that, in my lifetime, we solve the quantum gravity conundrum and finally arrive at the Theory of Everything. The Continuum Hypothesis, too, remains one of our species' enduring questions. I believe the one who solves the Vault may help push our species closer—toward a Theory of Everything, toward the edge of the Continuum itself—because disciplined reasoning, practiced and preserved, is a tool that endures when even memory fades.
As you step forward, the threshold stands open. May the Vault leave in you an echo of something you always longed to have.

—Aaditya Paudel

Acknowledgements

This book would not have been possible without my dearest—who, as I write this, is roasting sesame seeds for me.

I want to thank my family, my friends, and all my well-wishers for helping me get here.

Thank you to the giants on whose shoulders I stand.

To my professors and my gurus: thank you for shaping me into someone able to do what I do now.

To my students, thank you for accepting the knowledge I impart. A special thank-you to Miami University's precalculus students who attempted 'The Mirror Gate of Caldera'- one of the puzzles featured here, and got them right—an early validation of this work.

I'm deeply grateful to the platforms that supported my applied AI work—Outlier, Mercor, and Snorkel.

And to my clients, thank you for challenging me to craft better and better problems.

And lastly, I'd like to thank me—for my unwavering relentlessness.

Archivist's Prologue

Lore

Blood, bones, and skin will one day wither away. But ideas remain.
I seek to archive the ideas—and to forge the myth into reality.
Welcome to The Vault. Here, you may find solace. For it is no easy thing to tread the web of lemmas, proofs, symbols, and integers.
But tread you shall—so long as you choose. And enjoy you may, should you solve the problems presented within—correctly, at that.

—Neohm

Contents

How to Read the Vault

This Vault is built for slow reading. The lore sets the temperature; the puzzle sets the rules; the Vault Directive tells you what the lock will accept.

Structure. Each seal is self-contained in three pages:

- **Page A:** the seal's header and index frame.
- **Page B:** three contained boxes (Lore, Puzzle, Vault Directive).
- **Page C:** a single-page solution ending in a boxed final answer.

You can read straight through the corridors, or use the Domain Atlas to navigate by topic.

Domain Atlas

The Vault is arranged into corridors; this atlas maps each seal to its primary domain for navigation, not for difficulty. Each seal is self-contained; you may enter in any order.

No.	Domain	Seal
1	Probability & Inference	High-Card Poker: "Lower Than Six"
2	Probability & Inference	High-Card Poker: The Claimed Jack
3	Probability & Inference	The Missing Treasure
4	Probability & Inference	Two Clues, One Suspect
5	Logic & Deduction	The Jade Serpent
6	Logic & Deduction	The Violet Moon Rite
7	Logic & Constraints	The Clockwork Alibi Ledger
8	Logic & Constraints	The Gallery Grid of Half-True Plaques
9	Theorem Web (Mixed)	The Locksmith's Quintuple Code
10	Number Theory & Divisibility	The Divisibility & Factorization Scroll
11	Proof Audit (Mixed)	The Ledger of Five Seals
12	Linear Algebra & Determinants	The Five Seals of the Linear Vault
13	Combinatorics & Discrete Structures	Five Lemmas from the Combinatorics Vault
14	Symbolic Logic	$\Omega \uparrow \oplus$
15	Algebra & Symmetric Sums	The Oath of Two Echoes
16	Combinatorics & Parity	The Twin-Wick Corridor
17	Geometry (Coordinate)	The Mirror Gate of Caldera
18	Geometry (Construction)	The Brass Protractor Rite
19	Geometry (Locus & Ratios)	The Three Bead Readings
20	Combinatorics & Recurrence	The Staircase Registry
21	Games & Strategy	The Nine Lenses
22	Tiling & Invariants	The Talon Plates
23	Geometry & Graph Theory	The Mirror Lemma Key
24	Geometry & Inequalities	The Balancer's Verdict
25	Titan (Valuations & Theorem Web)	The Tangent-Wrought Five-Count Seal

I
Lanterns of Chance

Chance and information share a single hinge: a belief shifts when evidence arrives.
This section treats probability as narrative pressure–cards, rituals, and staged signals–then resolves each story with exact arithmetic and clean conditioning.

1

High-Card Poker: “Lower Than Six”

The Final Wager

This is a one-hand inference puzzle: a single spoken claim about a hidden card is a noisy signal, and you must update the hidden-card distribution before you can compute the chance of a strict win. It tests clean sample-space counting in a 52-card deck, Bayes conditioning under an explicit truth/lie rate, and careful treatment of “win” versus “tie.” The only required tools are elementary counting and conditional probability.

Lore

The Grand Archive Casino hosts a simple game: the higher card wins, and ties pay nothing.
Each table is sleeved in green felt, inlaid with walnut dark as old tea and a seam of obsidian that catches the light like a blade. The cards are linen-backed, worn smooth by centuries of hands. The chandeliers do not merely flicker; they *remember*. Here, a remark is never just a remark. It is a move.
Around the chamber hang paintings—twenty of them—none newer than the truths they suggest. The house permits ornament, but it forbids invention. Every detail in this room is placed on purpose, and every placement is recorded.

> *“A faithful replica of Van Gogh’s* Cafe Terrace at Night*. Van Gogh portrayed the reality of life through his painting—here, all descriptions are literal.”*

To **Cyrene**’s right, the cafe glows under an indigo sky. Three figures sit at a table near the center, faces unfinished, conversation lost to time. A glass catches a thin blade of gold. The frame is heavy gilt—polished, too perfect, and therefore not old. It is expensive in the way a mask is expensive.
To her left hangs something that does not bother with beauty. It is a panel, pigment laid straight onto rough grain, held in a cradle of dark wood. No signature. No gallery stamp. A circular table, mid-shuffle. No hands. An overturned glass. The shadows fall in quiet blocks, the kind that look, at a distance, like numbers.
At the bottom of that frame, etched into a narrow strip of brass—dulled by touch—is a line the house has not polished away:

> *Did you know? In over 10,000 observed games, comparative statements spoken* during play *are correct only one time in four.*

The dealer burns one card. The room goes still enough to hear the felt breathe.

Puzzle

High-card rules. Each player holds exactly one card from a standard 52-card deck. Ranks are ordered

$$2 < 3 < \cdots < 10 < J < Q < K < A.$$

Suits do not matter. **Cyrene** *wins* if and only if her rank is *strictly higher* than **Irin**’s rank. (A tie is not a win.)

Cyrene looks down at 8♣. Irin draws one card uniformly at random from the remaining 51 cards and keeps it hidden.

Irin makes exactly one comparative claim about his rank relative to 6: either “my card is lower than 6” or “my card is 6 or higher.” Cyrene hears Irin say:

“My card is lower than 6, Cyrene. Your move.”

Plaque rule. Treat the brass-plaque statistic as binding for this exact two-option claim.

Vault Directive

What is the probability that Cyrene wins given Irin’s claim? Give your answer as a percentage rounded to the nearest whole percent.

Solution

Let L be the event “Irin’s rank is below 6” (i.e. 2, 3, 4, 5), and let S be the event “Irin says ‘lower than 6’.” By the plaque rule,

$$P(S \mid L) = \frac{1}{4}, \qquad P(S \mid \neg L) = \frac{3}{4}.$$

Priors.
Cyrene holds 8♣, so Irin draws uniformly from the remaining 51 cards. There are 16 cards of ranks 2–5, hence

$$P(L) = \frac{16}{51}, \qquad P(\neg L) = \frac{35}{51}.$$

Bayes.

$$P(S) = \frac{1}{4} \cdot \frac{16}{51} + \frac{3}{4} \cdot \frac{35}{51} = \frac{121}{204}.$$

So

$$P(L \mid S) = \frac{(\frac{1}{4})(\frac{16}{51})}{\frac{121}{204}} = \frac{16}{121}, \qquad P(\neg L \mid S) = \frac{105}{121}.$$

Win chance.
If L occurs, then Irin holds a 2, 3, 4, or 5, so Cyrene’s 8 wins surely:

$$P(\text{win} \mid L, S) = 1.$$

If $\neg L$ occurs, Irin’s card lies among the 35 remaining cards with ranks $6, 7, 8, 9, 10, J, Q, K, A$ (excluding 8♣). Cyrene’s 8 beats only ranks 6 and 7, i.e. 8 winning cards, so

$$P(\text{win} \mid \neg L, S) = \frac{8}{35}.$$

Therefore,

$$P(\text{win} \mid S) = \frac{16}{121} \cdot 1 + \frac{105}{121} \cdot \frac{8}{35} = \frac{40}{121} \approx 0.33058.$$

Answer

33%

2

High-Card Poker: The Claimed Jack

The Price of a Name

This is a Bayesian inference puzzle where the likelihood of a claim is governed by a strict institutional protocol. A specific card is named, but its reliability depends on a historical ledger modified by “primer” marks and a mechanical wheel that enforces false identities. The problem tests your ability to translate narrative rules into exact probabilistic weights, handle the conditioning of a lie on the available sample space, and compute the shift between a baseline and a posterior winning probability.

Lore

In the Archive Casino, a named claim is treated as a financial instrument.
The high-card side duel is simple: one card each, higher rank takes the pot, ties return the stake without profit. What complicates the table is not the game, but the archive. The house maintains a ledger of named claims and will not permit a player to name a card unless the claim is governed by the published policy for that table.
A brass plate is bolted to the rail. It does not list probabilities; it lists governance.
First, the record:

> *Named-claim record (last 1200): truth:lie odds = 1:2.*

Second, the adjustment:

> *Before pricing a new name, enter the Primer: one Truth and one Lie.*

The table does not debate what the Primer means. The clerk amends the ledger accordingly and treats the amended fraction of Truth as the trust placed in the next named claim.
False names are regulated by mechanism rather than discretion. If a player is not speaking the card they hold, the dealer invokes the Wheel: a balanced device that selects a card-identity uniformly from all identities except the one actually held. The Wheel disregards what is already visible on the felt; it excludes only the card in hand.

Puzzle

High-card rules. A standard deck has 52 distinct cards. Each player holds exactly one card. Ranks are ordered

$$2 < 3 < \cdots < 10 < J < Q < K < A.$$

Suits do not matter. Higher rank wins; a tie is not a win.

Deal. Raven draws 7♣ and leaves it face-up. Sable draws one card uniformly at random from the remaining 51 cards and keeps it hidden. Before any reveal, Sable announces:

> "My card is $J\diamondsuit$."

Named-claim policy. Interpret the reliability of Sable's announcement, and the distribution of a false announcement, exactly as specified on the brass plate described in the Lore.

Vault Directive

By how many percentage points does Raven's probability of a strict win change after hearing the announcement, compared to the no-announcement baseline?
Report the magnitude as a whole-number percentage in the form $D\%$ (R) if the announcement reduces Raven's winning chance, or $D\%$ (I) if it increases it.

Solution

Let S be the event "Sable says $J\diamondsuit$," and A the event "Sable actually holds $J\diamondsuit$."

Baseline (no announcement). Among the 51 possible hidden cards, Raven's 7 wins exactly against ranks 2 through 6, i.e. 5 ranks $\times 4$ suits $= 20$ cards. Hence

$$W_0 = \frac{20}{51} \approx 0.3921569.$$

Trust level from the brass plate. Truth:lie odds $1 : 2$ over 1200 hands gives 400 truths and 800 lies. The Primer adds one truth and one lie, so the trust level for the next named claim is

$$p = \frac{401}{1202}.$$

Hearing $J\diamondsuit$. Since Raven holds 7♣,

$$P(A) = \frac{1}{51}, \qquad P(\neg A) = \frac{50}{51}.$$

Also

$$P(S \mid A) = p, \qquad P(S \mid \neg A) = (1-p) \cdot \frac{1}{51},$$

because in false-mode the Wheel selects uniformly among the 51 identities not equal to the card in hand (including identities already visible on the felt).
Bayes gives

$$P(\neg A \mid S) = \frac{P(S \mid \neg A)P(\neg A)}{P(S \mid \neg A)P(\neg A) + P(S \mid A)P(A)} = \frac{50(1-p)}{50(1-p) + 51p} = \frac{13350}{20167}.$$

Win after the announcement. Raven loses if A occurs since $J > 7$. If $\neg A$ occurs under S, Sable's actual card is uniform among the 50 cards excluding $J\diamondsuit$ and excluding Raven's 7♣. Raven wins against 20 of these cards, so

$$W_1 = P(\neg A \mid S) \cdot \frac{20}{50} = \frac{13350}{20167} \cdot \frac{2}{5} = \frac{5340}{20167} \approx 0.2647890.$$

Change in percentage points.

$$100(W_1 - W_0) \approx 26.47890 - 39.21569 \approx -12.73679.$$

The magnitude rounds to 13, and the sign is a reduction.

Answer

13% (R)

3

The Missing Treasure

Warmup • Base Rates • Posterior

This is a two-stage forensic puzzle: first, you must model a state-dependent sampling process (the “sock ritual”) to derive a precise base rate, and then you must apply that prior to a Bayesian likelihood comparison to identify which suspect possesses guilty knowledge.

Lore

They called it treasure only because it felt like one: velvet-wrapped, rain-chilled, still gritty at the seams.
Every drawer received a brittle one-use strip, but the drawer that mattered received a strip chosen by chance—because chance is easier to argue with than memory.

Puzzle

Five friends—**Raven**, **Athena**, **Quill**, **Kestrel**, and **Sable**—found a velvet-wrapped treasure while out walking. They agreed Raven would keep it overnight for safekeeping. Raven left the group at **6:00 PM** and put the treasure in one dresser drawer.

Raven sealed *every* dresser drawer with a one-use security strip. (A strip irreversibly shatters if that drawer is opened even slightly.)

The hidden-mark strip. Only the treasure drawer got a special strip whose underside fold carried *one* hidden mark: either a small **Wave** or a small **Knot**. The mark is invisible unless the strip shatters.
Which mark Raven used was decided by a ritual performed *before* sealing:

> Raven has a bag containing **5 pairs of socks**, each pair a different color (so 10 socks total). She draws **4 single socks** at random and holds them aside. If the 4 socks in hand contain any *complete color-pair* (both socks of some color), Raven discards that complete pair and immediately draws replacements (one sock per discarded sock) from the bag until 4 socks are in hand again.
>
> She repeats this discard-and-replace process until either:
>
> - the hand contains **4 socks of 4 different colors** (then a **Wave** strip is used), or
> - the bag becomes **empty** before 4 different colors are ever reached (then a **Knot** strip is used).

At **8:10 PM**, Raven checked the dresser in passing: all strips were intact.
At **8:40 PM**, Raven checked again and found *exactly one* strip shattered. The treasure was gone. The shattered strip's hidden mark was a **Wave**.

Raven did *not* tell anyone which mark appeared. The next day Raven questioned the other four friends. To keep the stories from changing midstream, Raven did the following:

- She asked each person, alone, to write on a slip of paper: *"Wave or Knot: which do you believe was under the shattered strip?"*
- Each person sealed their slip in wax *before* speaking further.
- Everyone's slip and statement below is **truthful**: if someone actually opened the drawer and saw the mark, they wrote what they saw; otherwise they wrote according to their usual habit described below.
- Raven had **no reason beforehand** to suspect one friend more than another.

When Raven opened the sealed slips, the slips read:

> **Athena's slip: Wave**.
> **Athena added:** "If I don't know, I always bet on whichever mark is *more likely* under your sock ritual."
>
> **Kestrel's slip: Knot**.
> **Kestrel added:** "If I don't know, I always bet on whichever mark is *less likely*. I like long shots."
>
> **Sable's slip: Wave**.
> **Sable added:** "If I don't know, I try to 'match the odds': I choose Wave with the *same frequency* that your ritual produces Wave."
>
> **Quill's slip: Wave**.
> **Quill added:** "If I don't know, I flip a fair coin: Wave on heads, Knot on tails."

Vault Directive

Identify the most likely thief.

Solution

1. The Evidence The shattered strip revealed a **Wave**.

- **The Thief:** Must write **Wave** (because they saw it and are compelled to be truthful about what they saw).
- **The Innocent:** Writes Wave or Knot based on their guessing strategy.

Kestrel wrote Knot. Since the thief would have written Wave, Kestrel is **innocent**. We must determine who among Athena, Sable, and Quill is most likely to be the thief.

2. The Base Rate (Sock Ritual) We calculate $p = P(\text{Wave})$, the probability of drawing 4 distinct colors. Total socks: 10 (5 pairs). Initial draw: 4 socks. Total combinations: $\binom{10}{4} = 210$.

- **Immediate Wave (4 different colors):** Choose 4 pairs from 5, then 1 sock from each.

$$P(\text{Wave}_1) = \frac{\binom{5}{4} \times 2^4}{210} = \frac{5 \times 16}{210} = \frac{8}{21}$$

- **One Pair (2 different colors + 1 pair):** Choose 1 pair to match, 2 pairs for the singles.

$$P(1\text{ Pair}) = \frac{\binom{5}{1} \times \binom{4}{2} \times 2^2}{210} = \frac{5 \times 6 \times 4}{210} = \frac{4}{7}$$

  Action: Discard the pair. We hold 2 different socks. Bag has 6 socks left (contains the mates for our held socks + 2 fresh pairs). To get a Wave, we must draw the two fresh colors (one from each new pair) from the 6 remaining socks.

$$P(\text{Success} \mid 1\text{ Pair}) = \frac{2 \times 2}{\binom{6}{2}} = \frac{4}{15}$$

$$p = \frac{8}{21} + \left(\frac{4}{7} \times \frac{4}{15}\right) = \frac{40}{105} + \frac{16}{105} = \frac{56}{105} = \frac{8}{15}.$$

Since $8/15 > 7/15$, Wave is the **more likely** outcome.

3. Suspect Probabilities We compare the likelihood of the observed slips under each suspect's guilt.

- **Athena (Guilty):** She writes Wave (thief). Sable guesses Wave ($p = 8/15$). Quill guesses Wave ($p = 1/2$).

$$L_A = 1 \times \frac{8}{15} \times \frac{1}{2} = \frac{4}{15} \approx 0.26$$

- **Sable (Guilty):** She writes Wave (thief). Athena guesses Wave (bets "more likely", $p = 1$). Quill guesses Wave ($p = 1/2$).

$$L_S = 1 \times 1 \times \frac{1}{2} = \frac{1}{2} = 0.50$$

- **Quill (Guilty):** He writes Wave (thief). Athena guesses Wave ($p = 1$). Sable guesses Wave ($p = 8/15$).

$$L_Q = 1 \times 1 \times \frac{8}{15} = \frac{8}{15} \approx 0.53$$

The likelihood is highest for Quill.

Answer

QUILL

4

Two Clues, One Suspect

Bayes • Induction • Invariants • Sums of Squares

A whodunit built from two dependent-looking "clues" whose numerical weights are themselves forged through three gates: an extremal interval invariant (frames), an induction-counting lemma (crossings), and a sum-of-two-squares certification via modular arithmetic (prime dial). Expect to translate structure into probabilities, handle a hidden mixture over imitation, and then fuse the evidence cleanly with Bayes to extract a single negative-integer score.

Lore

The Archivists call it *the Walk of Three Gates*.
Frames first: one wrong shadow spoils the rest. Crossings next: safe paths are counted, never guessed. Last, the Prime Dial: it turns only when a prime agrees to be two squares.
A bowl of coins waits on the Crucible table, bisected by a single scratched line. The Initiate is told to ignore the noise, and compute what remains after both coins fall.

Puzzle

Two suspects are considered for the theft: **Eld** and **Nyra**.

$$P(\text{Eld}) = \frac{2}{5}, \qquad P(\text{Nyra}) = \frac{3}{5}.$$

If Nyra is guilty, she may mirror Eld's style. Let H = "Nyra mirrors", with

$$P(H \mid \text{Nyra}) = \frac{1}{2}.$$

Gate I (frames). A *frame* is an axis-aligned rectangle with positive side lengths. A set $B_1, \dots, B_F$ is arranged so that

$$B_i \cap B_j \neq \varnothing \iff i \not\equiv j \pm 1 \pmod F,$$

and the thief always uses as many frames as possible (so F is maximal). The **Frame Coin** is flipped by choosing an unordered pair of distinct frames uniformly; let E = "the chosen pair is disjoint". Eld always uses the cyclic rule above. If Nyra is guilty: if H, she uses the same rule; if $\neg H$, she uses a *chain* rule in which the disjoint pairs are exactly $\{(B_1, B_2), \dots, (B_{F-1}, B_F)\}$.

Gate II (crossings). Using the same F, draw F lines in the plane, no two parallel and no three concurrent, and choose a point O on no line. A crossing X (intersection of two lines) is *clear* if the open segment (OX) meets at most $F - 3$ lines. Let C be the minimum number of clear crossings that must exist (this minimum is known to be sharp).

Gate III (prime dial). Set $P = 4C + 1$ (for the value of C above, P is prime), and define

$$W \equiv (2C)! \pmod P, \qquad 1 \le W \le \frac{P-1}{2}, \qquad Y = \frac{W^2 + 1}{P} \in \mathbb{Z}_{>0}.$$

The Prime Gate certifies that there exist nonnegative integers $g < h$ and m, n such that

$$P = g^2 + h^2, \qquad Y = m^2 + n^2, \qquad W = gm + hn.$$

The **Prime Coin** is flipped by choosing one of the P dial detents uniformly; let J = "the detent is dangerous", with

$$P(J \mid \text{Eld}) = \frac{h}{P}, \quad P(J \mid \text{Nyra}, \neg H) = \frac{g}{P}, \quad P(J \mid \text{Nyra}, H) = \frac{g+h}{P}.$$

Dependence. Assume E and J are independent conditional on Eld, and also independent conditional on (Nyra, H) and (Nyra, $\neg H$).

You observe *both* E and J.

Vault Directive

Let $P(\text{Eld} \mid E, J) = \frac{A}{B}$ in lowest terms. Compute $A - B$.

Solution

Roadmap. Write $EJ := E \cap J$. We compute $F \to C \to (P, W, Y) \to (g, h)$, then the likelihoods $P(EJ \mid \text{Eld})$ and $P(EJ \mid \text{Nyra})$ (Nyra is a $\frac{1}{2}/\frac{1}{2}$ mixture over H), and finish with Bayes.

Gate I: find F. Let $X_i = [a_i, b_i] = \text{proj}_x(B_i)$ and $Y_i = [c_i, d_i] = \text{proj}_y(B_i)$. If $B_i \cap B_j \neq \varnothing$ then $X_i \cap X_j \neq \varnothing$ and $Y_i \cap Y_j \neq \varnothing$; hence

$$B_i \cap B_{i+1} = \varnothing \Rightarrow (X_i \cap X_{i+1} = \varnothing) \text{ or } (Y_i \cap Y_{i+1} = \varnothing). \qquad (*)$$

Choose indices so that $a_1 = \max\{a_1, \dots, a_F\}$. Then for every $i \not\equiv 2, F \pmod F$, the rule forces $B_i \cap B_1 \neq \varnothing$, hence $a_1 \in X_i$; in particular $X_i \cap X_{i+1} \neq \varnothing$ for $3 \le i \le F-2$. Thus only the four cyclic pairs $X_1 \cap X_2$, $X_2 \cap X_3$, $X_{F-1} \cap X_F$, $X_F \cap X_1$ can be empty, and not all four can be (empty $X_1 \cap X_2$ and $X_2 \cap X_3$ would force $X_2 \cap X_{F-1} = \varnothing$, contradicting the required intersection $B_2 \cap B_{F-1} \neq \varnothing$). Hence $\#\{i : X_i \cap X_{i+1} = \varnothing\} \le 3$, and similarly $\#\{i : Y_i \cap Y_{i+1} = \varnothing\} \le 3$. By $(*)$, each of the F disjoint adjacent frame-pairs is witnessed by an empty X-adjacency or Y-adjacency, so $F \le 3 + 3 = 6$. A configuration with $F = 6$ exists, so the maximal value is $F = 6$.

Gate II: find C. Set $t := F - 3$. The Crossing-Map lemma gives at least $\frac{1}{2}(t+1)(t+2)$ clear crossings, and this bound is sharp. Induction spine: pick a line ℓ closest to O; choose $P \in \ell$ so (OP) meets no line. Along each ray of ℓ from P, the number of lines crossed by (OX) changes by at most 1 between consecutive intersection points, yielding at least $t+1$ clear crossings on ℓ; removing ℓ reduces t by 1. With $F = 6$, $t = 3$, so $C = \frac{1}{2}(4)(5) = 10$.

Gate III: compute P, W, Y, g, h. Now $P = 4C + 1 = 41$ (prime) and $W \equiv (2C)! \equiv 20! \pmod{41}$ with $1 \le W \le 20$. By Wilson,

$$40! \equiv -1 \pmod{41}, \qquad 40! = (20!)(21 \cdots 40) \equiv (20!)^2 \pmod{41},$$

so $W^2 \equiv -1 \pmod{41}$ and hence $Y = (W^2+1)/41 \in \mathbb{Z}_{>0}$. Compute

$$20! = \prod_{i=1}^{10} i(21-i) \equiv 20 \cdot 38 \cdot 13 \cdot 27 \cdot 39 \cdot 8 \cdot 16 \cdot 22 \cdot 26 \cdot 28 \pmod{41}.$$

Grouping,

$$(20 \cdot 39)(38 \cdot 27)(13 \cdot 28)(8 \cdot 16)(22 \cdot 26) \equiv 1 \cdot 1 \cdot (-5) \cdot 5 \cdot (-2) = 50 \equiv 9 \pmod{41},$$

so $W = 9$ and $Y = (81+1)/41 = 2$. Now $W^2 + 1 = PY$ gives $PY - W^2 = 1$. Applying the $(xy - z^2 = 1) \Rightarrow$ "two-squares" lemma (the Prime Gate's certificate), there exist $g < h$ and $m, n \ge 0$ with

$$P = g^2 + h^2, \qquad Y = m^2 + n^2, \qquad W = gm + hn.$$

Since $Y = 2$, we have $(m, n) = (1, 1)$, so $W = g + h$. With $g^2 + h^2 = 41$ and $g + h = 9$, we get $gh = 20$, hence $(g, h) = (4, 5)$.

Likelihoods: With $F = 6$, $P(E \mid \text{cyclic}) = \frac{F}{\binom{F}{2}} = \frac{2}{F-1} = \frac{2}{5}$, $\qquad P(E \mid \text{chain}) = \frac{F-1}{\binom{F}{2}} = \frac{2}{F} = \frac{1}{3}$.
Thus $P(E \mid \text{Eld}) = \frac{2}{5}$, $\quad P(E \mid \text{Nyra}, H) = \frac{2}{5}$, $\quad P(E \mid \text{Nyra}, \neg H) = \frac{1}{3}$, and

$$P(J \mid \text{Eld}) = \frac{h}{P} = \frac{5}{41}, \quad P(J \mid \text{Nyra}, \neg H) = \frac{g}{P} = \frac{4}{41}, \quad P(J \mid \text{Nyra}, H) = \frac{g+h}{P} = \frac{9}{41}.$$

Independence gives

$$P(EJ \mid \text{Eld}) = \frac{2}{5} \cdot \frac{5}{41} = \frac{2}{41},$$

$$P(EJ \mid \text{Nyra}) = \frac{1}{2}\left(\frac{2}{5} \cdot \frac{9}{41}\right) + \frac{1}{2}\left(\frac{1}{3} \cdot \frac{4}{41}\right) = \frac{9}{205} + \frac{2}{123} = \frac{37}{615}.$$

Bayes:

$$P(\text{Eld} \mid E, J) = \frac{\frac{2}{5} \cdot \frac{2}{41}}{\frac{2}{5} \cdot \frac{2}{41} + \frac{3}{5} \cdot \frac{37}{615}} = \frac{\frac{4}{205}}{\frac{4}{205} + \frac{111}{3075}} = \frac{60}{60 + 111} = \frac{20}{57}.$$

So $A = 20$, $B = 57$, hence $A - B = -37$.

Answer

-37

II
Council of Witnesses

Here, truth is constrained by protocol: who spoke first, who saw which seal, which statements can coexist.
These puzzles read like depositions and ledgers, but the solutions reduce every account to a consistent logical core.

5

The Jade Serpent

Logic • Knights & Knaves • Implication

This is a classic "liar logic" puzzle wrapped in a whodunit. You must determine the unique truth-values of four suspects' statements to identify the single culprit. The key lies in finding a self-contradiction within a suspect's own testimony to break the symmetry.

Lore

The Jade Serpent was lifted cleanly from its keep, leaving a perfect rectangle in the dust where its coils had rested.
Caldera's courtyard is spare: hedge, stone, and an open stretch between the iron gate and the plinth at the center. The gardeners keep that open ground lightly dusted with pale pumice so that even soft shoes leave a record.
Two records survive, both the kind the Vault trusts more than recollection.
Pinned under the display glass is the gardener's slip:

22:25 — plinth brushed; idol present. *22:35 — plinth brushed; idol absent.*

And beside it, a lantern-plate from the courtyard camera (the flame-log that timestamps each exposure):

22:31 — exposure log. The serpent is visible on the plinth.

At 22:31 a clerk combed the pumice smooth. At 22:35 the clerk found *exactly one* continuous trail: from the gate threshold to the plinth and back again—and no other footprints anywhere after the 22:31 smoothing.
Above the gate hangs the hearing rule, cut into brass:

One tongue is inverted. Every sentence from that mouth is false. All others speak only truth.

Puzzle

Exactly one of the four suspects is a *universal liar* (every one of their statements below is false). The other three suspects speak only truth. Exactly one suspect stole the jade serpent.
From the Lore box, the serpent is on the plinth at 22:31 and gone by 22:35, and there is exactly one post-22:31 trail from the gate to the plinth and back. For brevity, call this trail the *plinth-run*. The thief is the person who made the plinth-run.

Suspects {**Athena**, **Quill**, **Kestrel**, **Sable**} give these statements:

Athena:
(1) *"I left the courtyard at 22:30 and did not return."*
(2) Exactly one of these is true: *"Quill made the plinth-run"* and *"Kestrel made the plinth-run."*
(3) Sable is the universal liar *if and only if* my 22:30 story is true.

Quill:
(1) If Athena's 22:30 story is true, then Sable did not make the plinth-run.
(2) If I made the plinth-run, then Athena's 22:30 story is false.
(3) Exactly one of these is true: *"I made the plinth-run"* and *"Sable is the universal liar."*

Kestrel:
(1) Exactly one of these is true: *"Quill made the plinth-run"* and *"Athena's 22:30 story is true."*
(2) Sable is the universal liar *only if* I made the plinth-run.
(3) Quill did not make the plinth-run.

Sable:
(1) Athena's 22:30 story is false.
(2) Exactly one of these is true: *"I made the plinth-run"* and *"Quill made the plinth-run."*
(3) Athena's 22:30 story is false *if and only if* Kestrel made the plinth-run.

Vault Directive

Who stole the Jade Serpent?

Solution

Let S be the proposition “Athena’s 22:30 story is true.”
From the Lore box, there is exactly one plinth-run after 22:31, and the thief is the person who made that plinth-run.

Sable cannot be a truth-teller.
Assume Sable is *not* the universal liar, so Sable’s statements are true.
Then Sable(1) implies S is false. With S false, Sable(3) implies “Kestrel made the plinth-run” is true.
But if Kestrel made the (unique) plinth-run, then neither Sable nor Quill made it. That makes Sable(2) false (it claims exactly one of “Sable made the plinth-run” and “Quill made the plinth-run” is true).
This contradicts the assumption that Sable’s statements are true. Therefore Sable is the universal liar.

The thief is therefore:
Since Sable is the universal liar, Sable(1) is false, so S is true.
Also Sable(3) is false. In Sable(3), the left side “S is false” is false (since S is true). For a statement of the form “A if and only if B” to be false, A and B must differ. Thus “Kestrel made the plinth-run” must be true.
Since the plinth-run identifies the thief, Kestrel stole the jade serpent.

Answer

Kestrel

6

The Violet Moon Rite

Logic • Truth-Tables • Constraint Satisfaction

This puzzle applies a strict "One Lie Per Witness" constraint. Instead of evaluating every possible world, the optimal strategy involves identifying internal contradictions within a single suspect's testimony to collapse the possibilities.

Lore

On the Violet Moon, the monastery runs its rites by lanternlight and habit.
Four acolytes—Athena, Quill, Kestrel, Sable—were the only ones assigned to the inner corridor that night. At dawn, the reliquary was empty and the ash on the threshold showed too many careful steps to be an accident.
The prior refuses confessions taken in panic. Instead, the prior uses an old rule of discipline:

> *Each acolyte will make four claims. Exactly one claim from each is a lie.*

It is not mercy. It is arithmetic.

Puzzle

A relic called the *Violet Moon* is stolen.
Rules.

- Exactly one of the four suspects stole the Violet Moon.
- Each suspect makes **four** statements.
- For **each** suspect, **exactly three** of their statements are true and **exactly one** is false.

Whenever a suspect says "*exactly one of the following two claims is true*," interpret it literally: one is true and the other is false.

Four suspects—**Athena**, **Quill**, **Kestrel**, **Sable**—give the following statements. (In each line, "*X did it*" means "*X stole the Violet Moon*.")

Athena:
(1) Kestrel did it *if and only if* Sable did *not* do it.
(2) Quill did it *if and only if* Sable did *not* do it.
(3) Sable did it.
(4) Exactly one of these is true: "Athena did it" and "Quill did it."

Quill:
(1) Athena did it *if and only if* Kestrel did it.
(2) Kestrel did it *if and only if* Sable did it.
(3) Exactly one of these is true: "Kestrel did it" and "Sable did it."
(4) If Kestrel did it, then Quill did it.

Kestrel:
(1) If Quill did it, then Kestrel did *not* do it.
(2) Exactly one of these is true: "Quill did it" and "Sable did it."
(3) Quill did it *if and only if* Kestrel did *not* do it.
(4) If Kestrel did it, then Sable did *not* do it.

Sable:
(1) If Quill did it, then Sable did *not* do it.
(2) If Sable did it, then Quill did it.
(3) Exactly one of these is true: "Athena did it" and "Sable did it."
(4) Quill did it *if and only if* Kestrel did it.

Vault Directive

Who stole the Violet Moon?

Solution

Let A, Q, K, S denote the propositions "Athena did it," "Quill did it," "Kestrel did it," "Sable did it." Exactly one of A, Q, K, S is true.

Key pressure point (Quill).

Quill(2) says $K \leftrightarrow S$. Quill(3) says exactly one of K and S is true. Those two claims cannot both be true, so at least one of Quill(2), Quill(3) is false.

But Quill has *exactly one* false statement, so *exactly one* of Quill(2), Quill(3) is false, and Quill(1) and Quill(4) are true.

Case 1: Quill(2) is false and Quill(3) is true.

Then exactly one of K, S is true. Since Quill(4) is true, $K \Rightarrow Q$. If K were true, then Q would also be true, contradicting that there is exactly one thief. Hence K is false, so S is true.

So in this case, S (Sable) stole the Violet Moon.

Case 2: Quill(2) is true and Quill(3) is false.

Then $K \leftrightarrow S$ is true, so K and S have the same truth value. They cannot both be true (only one thief), so K and S are both false.

Now Quill(1) is true: $A \leftrightarrow K$. With K false, this forces A false. Thus the only remaining candidate for the (unique) thief would be Q true.

But if Q is true while K and S are false, then Athena(1) says $K \leftrightarrow \neg S$, i.e. false $\leftrightarrow$ true, which is false; Athena(3) says S, which is also false. So Athena would have at least two false statements, contradicting the Echo-Stamp rule.

Therefore Case 2 is impossible.

Conclusion.

Only Case 1 remains, so S is true: **Sable stole the Violet Moon.**

Answer

Sable

7

The Clockwork Alibi Ledger

Temporal Logic • Scheduling • Alibi Verification

This is a rigorous timeline puzzle. You must cross-reference travel times, mandatory wait periods, and mechanical timestamps to construct a minute-by-minute schedule for four custodians. The solution requires identifying the only suspect whose path is mathematically compatible with the crime window.

Lore

The Ledger captures time. It is a machine of brass and pressure, indifferent to testimony. At every threshold, a signet-plate demands a mark. Strike it, and the minute is burned into the record. Pass through, and the gate may hold you, the locking pins refusing to retract until the corridor decides you have waited long enough.
The Vault hums with internal rhythms: the Scriptorium press, the Furnace bell, the Gatehouse latch. These side-strips chatter only when a hand is present to work them. And deep in the center, the Reliquary door is timed by a spring that knows no mercy—it opens once, waits, and slams shut. Four custodians swear innocence. The clockwork swears otherwise.

Puzzle

The Vault has four relevant places: the *Scriptorium* S, the *Furnace* F, the *Gatehouse* G, and the *Reliquary* R.

Timing convention. All timestamps are exact *to the minute*. Travel times and gate delays are in whole minutes. You may wait anywhere, but you may not arrive earlier than the stated travel minima.

Travel times (door-to-door minima).

- Each Way: | | $G \leftrightarrow S$: 5 minutes | | $G \leftrightarrow F$: 5 minutes | | $S \leftrightarrow F$: 4 minutes | |
- *Whisper Stair* (down/up): $S \rightarrow R$ takes 4 minutes, $R \rightarrow S$ takes 6.
- *Ash Duct* (down/up): $F \rightarrow R$ takes 3 minutes, $R \rightarrow F$ takes 5.
- Main hall: $G \leftrightarrow R$ takes 7 minutes each way.

How the Ledger works. To *enter* S, F, or G, a custodian must press their own signet on the outside plate; the Ledger prints the minute of entry. Exiting does *not* print anything. The Gatehouse has no such delay.
Two wing-gates have pressure seals:

- After entering S at minute t, you cannot exit S until minute $t + 6$.
- After entering F at minute t, you cannot exit F until minute $t + 4$.

Three clockwork side-strips. These strips are purely mechanical and cannot be faked. A stamp at minute t means the action was performed during minute t, and only someone *inside* that room during minute t could perform it:

- The Furnace safety bell can be silenced *only from inside* F; it stamped **SILENCED 21:22** and **SILENCED 21:27**.
- The Scriptorium press can be stopped *only from inside* S; it stamped **STOPPED 21:24**.
- The Gatehouse shutter latch can be reset *only from inside* G; it stamped **RESET 21:30**.

The locked Reliquary. The Reliquary is a true locked room (one door, no other exit). Its bolt stamps the minute when the door *first opens* and the minute when it *finally shuts*. Once opened, a spring holds the door open for *exactly three minutes*; it cannot be shut early.
A dent made the last digit hard to read; the archivist is certain it was *either* 3 *or* 5. So the bolt times were *either*

OPEN 21:23 and SHUT 21:26 or OPEN 21:25 and SHUT 21:28.

A thin ash layer inside shows only *one* person crossed the threshold during that single opening.
Inside, the coffer requires *two full minutes* of continuous cranking to open; it cannot be completed outside the three-minute opening.

Ledger entries (complete for 21:15–21:35).

- 21:15 — Aegis entered S.
- 21:16 — Athena entered F.
- 21:19 — Quill entered S.
- 21:20 — Kestrel entered G.
- 21:33 — Aegis entered F.
- 21:33 — Athena entered G.
- 21:34 — Quill entered F.
- 21:35 — Kestrel entered S.

Vault Directive

Who is the culprit (thief)?

Solution

Case 1: OPEN 21:23, SHUT 21:26.
This window lasts three minutes, so to complete a continuous two-minute crank, the entrant must be inside R no later than 21:24.

- **Aegis** entered S at 21:15 and cannot exit until 21:21. The fastest route to R is $S \to R$ (Whisper Stair) in 4 minutes, so Aegis's earliest arrival time is 21:25. Too late.
- **Quill** entered S at 21:19 and is sealed in until 21:25, so Quill cannot reach R by 21:24.
- **Kestrel** entered G at 21:20; the fastest $G \to R$ is 7 minutes, so Kestrel's earliest arrival time is 21:27. Too late.
- **Athena** is the only one who could, in principle, reach R by 21:24 (Athena entered F at 21:16 and can leave F from 21:20 onward; $F \to R$ takes 3 minutes).

However, the Furnace safety bell stamped **SILENCED 21:22**, and only someone *inside* F during minute 21:22 can do that. Under the ledger list, no one besides Athena can possibly be in F by 21:22.
So Athena must be in F at 21:22, which means Athena cannot leave F early enough to be in R by 21:24. Therefore no one can both satisfy the 21:22 Furnace stamp and still complete the two-minute crank within the 21:23–21:26 Reliquary opening. Thus **Case 1 is impossible**.

Case 2: OPEN 21:25, SHUT 21:28.
Now to complete two continuous minutes of cranking, the entrant must be inside R by 21:26.

- **Quill**: sealed in S until 21:25, then needs 4 minutes to descend $S \to R$, so earliest arrival is 21:29. Impossible.
- **Kestrel**: from G at 21:20, earliest $G \to R$ arrival is 21:27, leaving at most one minute before SHUT 21:28. Not enough for a two-minute crank.
- **Athena**: Athena must be inside F at **21:27** for the second **SILENCED 21:27** stamp, so Athena cannot also be the lone person in R for two continuous minutes within 21:25–21:28.
- **Aegis**: Aegis entered S at 21:15 and may first exit at 21:21. Taking the Whisper Stair down (4 minutes) puts Aegis at R at 21:25. Aegis can crank continuously for two full minutes (e.g., 21:25–21:27) within the three-minute opening.
  After the door shuts at 21:28, taking the Ash Duct up ($R \to F$ is 5 minutes) brings Aegis to F at 21:33, matching the ledger entry **21:33 — Aegis entered** F.

So **Aegis is the only possible entrant** in the only possible Reliquary window.

Consistency snap-fit (why the rest of the stamps align).
With Case 2 fixed and Aegis committed to R during 21:25–21:28:

- The Scriptorium press stamped **STOPPED 21:24**. Aegis cannot be in S at 21:24 (Aegis must have left by 21:21 to reach R at 21:25), so the only person who can be inside S at 21:24 is **Quill**, who entered S at 21:19 and is sealed in until 21:25.
- The Gatehouse latch stamped **RESET 21:30**. Kestrel entered G at 21:20 and has no seal preventing Kestrel from remaining inside until 21:30; then $G \to S$ takes 5 minutes, yielding Kestrel's entry **21:35 — Kestrel entered** S.
- Athena must be in F at both **21:22** and **21:27**. Since Athena later entered G at 21:33 and $F \to G$ takes 5 minutes, Athena must leave F at 21:28 (immediately after the 21:27 silencing) to arrive and enter G at 21:33.

All mechanisms agree on a single culprit.

Answer

Aegis

8

The Gallery Grid of Half-True Plaques

Logic Grid • Deduction • Unique Matching

A logic-grid gallery with half-true plaques. Spatial constraints, time ordering, and five XOR inscriptions force a unique configuration.

Lore

Five niches, five wards, five keepers.
Each alcove bears a two-line plaque: one sentence in brass, one in stone. The Vault was built by skeptics, so every plaque lies once and tells truth once.
Tonight, the gallery is not merely locked. A seized launch-capsule sits behind the eastern wall, and its countermand pad accepts *five digits*—no more—before the failsafe seals.
The ledger that should have named the relics is scraped clean. Only the half-true plaques remain, and the corridor will not forgive a single contradiction.
"Trust the grid. Then trust the order."

Puzzle

Five niches lie in a straight row from west to east, labeled I, II, III, IV, V. Each niche holds exactly one artifact, placed by exactly one custodian, at exactly one stamped time, sealed by exactly one ward. Each artifact, custodian, time, and ward is used exactly once.

Artifacts: Obsidian Spindle; Ivory Lark; Jade Censer; Copper Astrolabe; Ashglass Urn.
Custodians: Aegis; Athena; Kestrel; Quill; Sable.
Times: 21:11; 21:14; 21:17; 21:20; 21:23.
Wards: Ash; Brass; Mirror; Salt; Thread.

Archivist's fixed notes (all true).

1. The Mirror-warded niche is immediately to the west of the Ash-warded niche, and the Ash ward is on an *end* niche (I or V).
2. The Thread ward is not on an end niche.
3. The Mirror ward was set at the latest time.
4. The Brass ward was set exactly three minutes after the Ash ward.
5. The Thread ward was set at a time *strictly* between the Brass ward's time and the Mirror ward's time (so Brass $<$ Thread $<$ Mirror).
6. The Obsidian Spindle is the artifact sealed with Salt.
7. The Obsidian Spindle sits somewhere west of the Jade Censer.
8. The Jade Censer is *not* in niche IV, and it is in a niche adjacent to the Copper Astrolabe.
9. The Ashglass Urn sits somewhere west of the Ivory Lark, and the Ivory Lark was placed earlier than the Jade Censer.
10. Exactly one of these two statements is true: *Sable carried the Copper Astrolabe*; *Kestrel carried the Ivory Lark*.
11. If Quill did *not* carry the Obsidian Spindle, then Quill's time was 21:14.
12. Sable did not use the Mirror ward and did not use the Thread ward.
13. In the signet registry, the names sort alphabetically as Aegis $\prec$ Athena $\prec$ Kestrel $\prec$ Quill $\prec$ Sable. The custodian who set the Mirror ward comes *later* in that order than the custodian who set the Thread ward.

The five plaques (each plaque has exactly one true line).

1. **Plaque One:**
   - The Mirror ward stands immediately to the right of the Brass ward.
   - Aegis worked the middle niche (III).
2. **Plaque Two:**
   - Athena set the Salt ward.
   - The Ash ward was set at 21:20.
3. **Plaque Three:**
   - The Jade Censer is not in an end niche (I or V).
   - Quill carried the Ashglass Urn.
4. **Plaque Four:**
   - The Thread ward is in niche II.
   - The Copper Astrolabe is in niche V.
5. **Plaque Five:**
   - The Ivory Lark's time was exactly six minutes earlier than the Thread ward's time.
   - Aegis carried the Ivory Lark.

Countermand protocol (how to extract the 5-digit PIN).
After you determine the unique full assignment, order the *five placements by time* from earliest to latest. Let $r(\cdot)$ be the alphabetical rank from Note 13:

$$r(\text{Aegis}) = 1,\ r(\text{Athena}) = 2,\ r(\text{Kestrel}) = 3,\ r(\text{Quill}) = 4,\ r(\text{Sable}) = 5.$$

For each placement at time $21{:}mm$ by custodian C, form one digit as the *units digit* of $mm + r(C)$. Concatenate the five digits (earliest to latest) to obtain the PIN.

Vault Directive

Determine the five-digit countermand PIN. (You will need to deduce the complete assignment of artifact, custodian, time, and ward to each niche.)

Solution

From Note 1, Ash is an end niche and Mirror is immediately to its west, so

$$V = \text{Ash}, \qquad IV = \text{Mirror}.$$

By Note 3, $t(IV) = 21{:}23$. Notes 4–5 give $t(\text{Brass}) = t(\text{Ash}) + 3$ and $t(\text{Brass}) < t(\text{Thread}) < 21{:}23$, so $t(\text{Ash}) \in \{21{:}11, 21{:}14\}$.
Plaque 2 is half-true. If $t(\text{Ash}) = 21{:}20$ then Brass would be 21:23, leaving no time strictly between Brass and Mirror, contradicting Note 5; hence the second line of Plaque 2 is false and the first is true: **Athena set Salt.** By Note 6, Salt seals the Obsidian Spindle, so Athena carried the Spindle.
By Note 11, Quill did not carry the Spindle, hence $t(\text{Quill}) = 21{:}14$. If $t(\text{Ash}) = 21{:}11$, then $t(\text{Brass}) = 21{:}14$ and Quill would be the Brass custodian; under Notes 7–9 this forces the Jade Censer into niche IV, contradicting Note 8 (Jade $\neq IV$). Therefore

$$t(\text{Ash}) = 21{:}14, \quad t(\text{Brass}) = 21{:}17, \quad t(\text{Thread}) = 21{:}20, \quad t(\text{Salt}) = 21{:}11,$$

and in particular Quill is the Ash custodian in niche V.
Plaque 4 is half-true. If the Copper Astrolabe were in V, then by Note 8 the Jade Censer would be adjacent to V and hence in IV, again contradicting Note 8. Thus the second line of Plaque 4 is false and the first is true:

$$II = \text{Thread}.$$

Plaque 1 is half-true and IV = Mirror. Hence "Mirror immediately right of Brass" forces Brass in III, so the other line is false:

$$III = \text{Brass}, \qquad \text{Aegis} \neq III.$$

By Note 12, Sable used neither Mirror nor Thread, so Sable is not in IV or II. With I and V occupied by Athena and Quill, this forces

$$\text{Sable in } III.$$

Plaque 5 is half-true. Since $t(\text{Thread}) = 21{:}20$, its first line asserts $t(\text{Ivory}) = 21{:}14$, i.e. Ivory lies in niche V. If instead the second line were true (Aegis carried Ivory), then Ivory would not be in V, contradicting the time forced by Note 9 together with the ward-time skeleton already fixed. Hence the second line is false and the first is true:

$$V = \text{Ivory Lark}.$$

So Quill carried the Ivory Lark, and in Note 10 the clause "Kestrel carried the Ivory Lark" is false; therefore the other clause is true:

$$\text{Sable carried the Copper Astrolabe}.$$

Thus niche III holds the Copper Astrolabe.
By Note 8, the Jade Censer is adjacent to Copper (in III) and not in IV, so

$$II = \text{Jade Censer}, \qquad IV = \text{Ashglass Urn}.$$

The remaining custodians for II and IV are Aegis and Kestrel. By Note 13, the Mirror custodian (niche IV) is later alphabetically than the Thread custodian (niche II), so

$$II = \text{Aegis}, \qquad IV = \text{Kestrel}.$$

Therefore the unique assignment is

Niche	Artifact	Custodian	Time	Ward
I	Obsidian Spindle	Athena	21:11	Salt
II	Jade Censer	Aegis	21:20	Thread
III	Copper Astrolabe	Sable	21:17	Brass
IV	Ashglass Urn	Kestrel	21:23	Mirror
V	Ivory Lark	Quill	21:14	Ash

PIN. Chronological order: 21:11(Athena), 21:14(Quill), 21:17(Sable), 21:20(Aegis), 21:23(Kestrel).
Units digit of minute+rank: $11{+}2 \to 3,\ 14{+}4 \to 8,\ 17{+}5 \to 2,\ 20{+}1 \to 1,\ 23{+}3 \to 6$, hence $\boxed{38216}$.

Answer

38216

9

The Locksmith's Quintuple Code

Forensics • Theorem Web • Five-Digit Lock

This is a chain-reaction puzzle where the output of one mathematical sub-problem becomes the input parameter for the next. You must navigate Algebra, Graph Theory, Linear Algebra, and Combinatorics to derive the full code.

Lore

The Locksmith's door has a pin-code style key. It takes five dials—five small problems—each one stamped into bronze with a promise:

> *Turn me correctly and I will not argue with the next.*

Each dial yields a single digit. Together they form the code that releases the latch.

Puzzle

Five dials (I–V) each output one decimal digit. Let those digits be d_1, d_2, d_3, d_4, d_5. The Vault code is the 5-digit integer $\overline{d_1 d_2 d_3 d_4 d_5}$.

Dial I (Free-term collar). A collar of integer index $a \geq 1$ used for $n \geq 1$ full torque cycles stamps the difference

$$\Delta = a^{n+1} - (a+1)^n.$$

The sleeve shows $\Delta = 223137$. Dial I outputs

$$d_1 = n.$$

Dial II (Marriage minimum). Let $N = 2^{d_1+1}$. At a party there are N boys and N girls. Each boy knows at least m girls, and each girl knows at least m boys. Define $m_*(N)$ to be the *smallest* integer m such that a complete pairing (a matching of all boys to distinct acquainted girls) is guaranteed for *every* such acquaintance pattern. Dial II outputs

$$d_2 = m_*(N).$$

Dial III (The pegboard). On an $N \times N$ grid with rows and columns indexed $1, 2, \ldots, N$, a peg is present in cell (i, j) iff $i + j$ is even. Let P be the number of ways to choose N pegs so that no two chosen pegs lie in the same row or the same column. Dial III outputs

$$d_3 \equiv P \pmod{10}.$$

Dial IV (Commutator strip). Let $t = d_2 - 1$. Let X and B_0 be $t \times t$ real matrices, and define

$$B_k = B_{k-1}X - XB_{k-1} \qquad (k \geq 1).$$

Assume $B_{t^2} = X$. Dial IV outputs

$$d_4 = t^2.$$

Dial V (Factorial determinant). Let $M = d_4 + 1$. Define a sequence $(u_n)_{n\geq 0}$ by $u_0 = u_1 = u_2 = 1$ and, for every $n \geq 0$,

$$\det\begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n!.$$

Dial V outputs

$$d_5 \equiv u_M \pmod{10}.$$

Vault Directive

Determine the 5-digit code $\overline{d_1 d_2 d_3 d_4 d_5}$.

Solution

We compute the dials in order.

Dial I. If $a^{n+1} - (a+1)^n = 223137$ with integers $a, n \geq 1$, then a is an integer root of $f(x) = x^{n+1} - (x+1)^n - 223137$, so $a \mid f(0) = 223138 = 2 \cdot 31 \cdot 59 \cdot 61$. Modulo 3, $223137 \equiv 0$, forcing $a \equiv 1 \pmod 3$ and n even. Modulo 4, a must be odd and then $a \equiv 1 \pmod 4$. Among odd divisors of $31 \cdot 59 \cdot 61$, the congruences $a \equiv 1 \pmod 3$ and $a \equiv 1 \pmod 4$ force $a = 61$. Then

$$61^{n+1} - 62^n = 223137.$$

Check $n = 2$: $61^3 - 62^2 = 226981 - 3844 = 223137$. If $n \geq 4$ even, then $62^n \equiv 0 \pmod 8$, so $61^{n+1} \equiv 223137 \equiv 1 \pmod 8$, but $61 \equiv 5 \pmod 8$ and $n+1$ odd gives $61^{n+1} \equiv 5 \pmod 8$, contradiction. Hence $n = 2$, so $d_1 = 2$.

Dial II. $N = 2^{d_1+1} = 2^3 = 8$. Sharpness: for even N, split boys into B_1, B_2 of size $N/2$ and girls into G_1 of size $N/2 - 1$ and G_2 of size $N/2 + 1$. Connect every boy to every girl in G_1, and connect B_1 also to all girls in G_2. Then every vertex has degree $\geq N/2 - 1$, but B_2 has neighborhood G_1 of size $N/2 - 1$, so Hall fails and no perfect matching exists. Hence $m_*(N) \geq N/2$. Hall's condition shows $m_*(N) \leq N/2$: if every vertex has degree $\geq N/2$, then for any set X of boys, either $|X| \leq N/2$ and $|N(X)| \geq N/2 \geq |X|$, or $|X| > N/2$ and then $N(X)$ is all girls (otherwise some girl has degree $< N/2$). Thus $m_*(8) = 4$, i.e. $d_2 = 4$.

Dial III. The even-parity cells form two 4×4 blocks after permuting rows/cols by parity. Choosing 8 nonattacking pegs means choosing a permutation in each block, so $P = (4!)^2 = 576$, hence $d_3 \equiv 6 \pmod{10}$, i.e. $d_3 = 6$.

Dial IV. $t = d_2 - 1 = 3$, so $t^2 = 9$. Let V be the vector space of $t \times t$ matrices ($\dim V = t^2$). With $T(Y) = YX - XY$, we have $B_k = T^k(B_0)$. The $t^2 + 1$ vectors $B_0, \ldots, B_{t^2}$ are dependent; take a dependence with smallest index k having nonzero coefficient. Applying T repeatedly shifts the dependence and forces B_{t^2} to be a linear combination of $B_{t^2+1}, B_{t^2+2}, \ldots$. But $B_{t^2} = X$ and $B_{t^2+1} = [X, X] = 0$, hence all later B's are 0, so the only possibility is $B_{t^2} = 0$, i.e. $X = 0$. Therefore Dial IV outputs $d_4 = t^2 = 9$.

Dial V. $M = d_4 + 1 = 10$. The determinant rule is $u_{n+3}u_n - u_{n+2}u_{n+1} = n!$, which yields the closed form $u_n = (n-1)(n-3)\cdots$ (same-parity descending product), hence $u_{10} = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 945$, so $d_5 \equiv 5 \pmod{10}$, i.e. $d_5 = 5$.
Thus the code is $d_1d_2d_3d_4d_5 = 24695$.

Answer

24695

III
Scroll of Lemmas

Inside the lemma vault, arithmetic scrolls and algebraic webs whisper of sums, products, parity, and structure. Each seal demands the precise statement of a hidden theorem and the proof that no other path fits.

10

The Divisibility & Factorization Scroll

Warmup • Möbius • Ramanujan • Cyclotomy

This is a "True/False" gauntlet covering five classic identities in multiplicative number theory. You must verify properties of the Möbius function, Dirichlet convolutions, and cyclotomic polynomials to derive the correct binary string.

Lore

A scroll of copper foil lies on the lectern, etched with five claims.
The Archivist does not ask for proofs here.
"Only mark what is true. The Vault reads patterns, not prose."

Puzzle

For a positive integer n, let:

- $\mu(n)$ be the Möbius function,
- $\varphi(n)$ be Euler's totient,
- $\sigma(n) = \sum_{d|n} d$ be the sum-of-divisors function,
- $\mathrm{rad}(n) = \prod_{p|n} p$ be the product of distinct primes dividing n.

For integers $q \geq 1$ and n, define the Ramanujan sum

$$c_q(n) = \sum_{\substack{1 \leq k \leq q \\ (k,q)=1}} e^{2\pi i k n/q}.$$

For $n \geq 1$, define the nth cyclotomic polynomial

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ (k,n)=1}} \left(x - e^{2\pi i k/n}\right).$$

Consider the following statements.

(1) (Möbius–radical transform.) For all $n \geq 1$,

$$\sum_{d|n} \mu(d)\ \mathrm{rad}(n/d) = \varphi(n) \quad \Longleftrightarrow \quad n \text{ is squarefree.}$$

(2) (Möbius inversion for σ.) For all $n \geq 1$,

$$\sum_{d|n} \mu(d)\, \sigma(n/d) = n.$$

(3) (Radical as an exponent-weighted divisor product.) For all $n \geq 1$,

$$\prod_{d|n} d^{\mu(n/d)} = \mathrm{rad}(n).$$

(4) (Ramanujan divisor sum.) For all $q \geq 1$ and integers n,

$$\sum_{d|q} c_d(n) = \begin{cases} q, & q \mid n, \\ 0, & q \nmid n. \end{cases}$$

(5) (Cyclotomic Möbius product.) For all $n \geq 1$,

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

Vault Directive

Let $t_i = 1$ if statement (i) is true and $t_i = 0$ if it is false. Output the 5-bit string $t_1 t_2 t_3 t_4 t_5$.

Solution

1. **True.** Let $R = \mathrm{rad}(n)$. Since $\mu(d) = 0$ unless d is squarefree, only divisors $d \mid R$ contribute, and for such d we have $\mathrm{rad}(n/d) = R/d$. Hence

$$\sum_{d|n} \mu(d)\ \mathrm{rad}(n/d) = \sum_{d|R} \mu(d)\, \frac{R}{d} = R \prod_{p|R} \left(1 - \frac{1}{p}\right) = \varphi(R).$$

So the left-hand side equals $\varphi(R)$ always, and it equals $\varphi(n)$ iff $\varphi(R) = \varphi(n)$. But $\varphi(n) = \varphi(R) \prod_{p^a \| n} p^{a-1}$, so equality holds iff every $a = 1$, i.e. n is squarefree.

2. **True.** This is the Dirichlet convolution identity $\mu * \sigma = \mathrm{id}$, i.e.

$$\sum_{d|n} \mu(d)\, \sigma(n/d) = n.$$

3. **False.** Take $n = 12$. Then the product is

$$\prod_{d|12} d^{\mu(12/d)} = 2^{\mu(6)}\, 4^{\mu(3)}\, 6^{\mu(2)}\, 12^{\mu(1)} = 2 \cdot 4^{-1} \cdot 6^{-1} \cdot 12 = 1,$$

while $\mathrm{rad}(12) = 2 \cdot 3 = 6$.

4. **True.** Let $\zeta_q = e^{2\pi i/q}$. The qth roots of unity split into disjoint sets of primitive dth roots as d ranges over divisors of q. Therefore

$$\sum_{d|q} c_d(n) = \sum_{\zeta^q=1} \zeta^n = \begin{cases} q, & q \mid n, \\ 0, & q \nmid n, \end{cases}$$

since it is the full geometric sum over qth roots.

5. **True.** From the factorization $x^n - 1 = \prod_{d|n} \Phi_d(x)$ and Möbius inversion on the divisor lattice,

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

Thus $(t_1, t_2, t_3, t_4, t_5) = (1, 1, 0, 1, 1)$ and the Scroll Output is

$$\boxed{11011}.$$

Answer

11011

11

The Ledger of Five Seals

Mixed • Theorems • Truth Vector

Five independent claims are sealed into a single lock: only the correct truth values form the five-bit signature. Each statement is self-contained, but the door opens only if you classify all five correctly.

Lore

The Ledger of the Five Seals is written in a cramped hand, the kind of handwriting that only appears when someone is trying to make a rule survive an emergency.
It lists five charms. Each charm can be set to one of two states. The margin note—underlined twice—says:

> *Only the full five-bit signature opens the door. Nothing else.*

The clerk asks for the signature itself and nothing more.

Puzzle

For each of the following five statements, assign a truth value: assign 1 if the statement is true and 0 if it is false. Express your final answer as a 1×5 matrix

$$T = [t_A, t_B, t_C, t_D, t_E].$$

A. No-new-prime claim for odd prime exponents.
Let m and n be coprime positive *odd* integers with $m > n$, and let $k \geq 5$ be an *odd prime*. *Claim:* For every prime r dividing $m^k - n^k$, there exists a *proper divisor* j of k (so $1 \leq j < k$ and $j \mid k$) such that

$$r \mid m^j - n^j.$$

B. Cubic non-residue inverse-sum.
Let p be a prime with $p \equiv 1 \pmod 3$ and $p > 7$. Define

$$CR_p = \{a \in \{1, 2, \dots, p-1\} : \ a \text{ is } \textit{not} \text{ a cubic residue modulo } p\}.$$

(For $a \in \{1, \dots, p-1\}$, write a^{-1} for the multiplicative inverse of a modulo p.) *Claim:*

$$\sum_{a \in CR_p} a^{-1} \equiv 0 \pmod p.$$

C. Triangle-free graph edge bound.
Let $G = (V, E)$ be a simple graph on n vertices. *Claim:* If G is triangle-free, then

$$|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

D. Subset pairing symmetry lemma (interior k only).
Let $n \geq 3$ and let S be a finite set with $|S| = n$, and let $f : S \to \mathbb{R}$. For each k with $0 \leq k \leq n$, define

$$F(k) = \sum_{\substack{T \subseteq S \\ |T| = k}} \sum_{x \in T} f(x).$$

Claim:

$$\big(\forall\, k \in \{1, 2, \dots, n-1\},\ F(k) = F(n-k)\big) \iff \left(\sum_{x \in S} f(x) = 0\right).$$

E. Ultimate monotonic subsequence lemma.
Let n be a positive integer and let $A = (a_1, a_2, \dots, a_{n^2+1})$ be any sequence of $n^2 + 1$ distinct real numbers. Let $\ell_{\text{inc}}(A)$ and $\ell_{\text{dec}}(A)$ denote the lengths of the longest strictly increasing and strictly decreasing subsequences, respectively. *Claim:*

$$\max\{\ell_{\text{inc}}(A), \ell_{\text{dec}}(A)\} \geq n + 1.$$

Vault Directive

Output the 5-bit signature $t_A t_B t_C t_D t_E$.

Solution

A. False. Since k is an odd prime, its only proper divisor is $j = 1$, so the claim reduces to: *every prime divisor r of $m^k - n^k$ must divide $m - n$.*
Counterexample: take $m = 3$, $n = 1$, $k = 5$ (coprime, odd, and k is an odd prime). Then

$$3^5 - 1^5 = 243 - 1 = 242 = 2 \cdot 11^2.$$

The prime 11 divides $3^5 - 1^5$, but $11 \nmid (3 - 1) = 2$. So the claim fails.

B. True. Let $G = \mathbb{F}_p^\times$, a cyclic group of order $p - 1$. Since $p \equiv 1 \pmod 3$, the set of nonzero cubes forms a subgroup $H \leq G$ of index 3, with

$$|H| = \frac{p-1}{3} > 1 \quad (\text{since } p > 7).$$

Thus CR_p is exactly the union of the two nontrivial cosets of H.
First, $\sum_{h \in H} h \equiv 0 \pmod p$: the elements of H are precisely the roots in $\mathbb{F}_p$ of $x^{|H|} - 1$, whose $x^{|H|-1}$ coefficient is 0, so the sum of its roots is 0. Hence for any $g \in G$, the coset gH sums to

$$\sum_{x \in gH} x = g \sum_{h \in H} h \equiv 0 \pmod p.$$

Therefore the sum of all elements of CR_p is $0 \pmod p$.
Now observe that inversion permutes CR_p: if $a \notin H$, then $a \in gH$ or $a \in g^2H$ for some $g \notin H$, and $a^{-1} \in (gH)^{-1} = g^{-1}H = g^2H$, still outside H. So $\{a^{-1} : a \in CR_p\} = CR_p$, and

$$\sum_{a \in CR_p} a^{-1} \equiv \sum_{a \in CR_p} a \equiv 0 \pmod p.$$

C. True. This is Mantel's theorem: among triangle-free graphs on n vertices, the maximum number of edges is attained by the complete bipartite graph with parts as equal as possible, giving $\lfloor n^2/4 \rfloor$.

D. True. For $k \geq 1$, each $x \in S$ appears in exactly $\binom{n-1}{k-1}$ subsets $T \subseteq S$ of size k, so

$$F(k) = \binom{n-1}{k-1} \sum_{x \in S} f(x).$$

In particular,

$$F(1) = \sum_{x \in S} f(x), \qquad F(n-1) = \binom{n-1}{n-2} \sum_{x \in S} f(x) = (n-1) \sum_{x \in S} f(x).$$

If $F(1) = F(n-1)$ and $n \geq 3$, then

$$\sum f(x) = (n-1) \sum f(x) \implies (n-2) \sum f(x) = 0 \implies \sum f(x) = 0.$$

Conversely, if $\sum f(x) = 0$, then $F(k) = 0$ for every $k \geq 1$, so $F(k) = F(n-k)$ holds for all $k = 1, \ldots, n-1$.

E. True. This is exactly the Erdős–Szekeres monotone subsequence theorem for sequences of length $n^2 + 1$ of distinct real numbers.

Therefore,

$$T = [0, 1, 1, 1, 1].$$

Answer

01111

12

The Five Seals of the Linear Vault

Determinants • Interlocked Gates • The Vault Exponent

This is a chain of determinant problems where parameters cascade from one matrix to the next. You must compute the 2-adic valuation (the power of 2) of the final determinant.

Lore

At the last door of the Linear Vault, Eryn finds a mercy carved into the lintel:

> *Do not speak the whole verdict. Speak only its height.*

Five seals must still be set—staircase, mirror, choir, black–white wall, and the imaginary bolt— but the tribute is not the determinant itself.
The Vault asks only for how many times the verdict can be halved before an odd number remains.

Puzzle

Athena must forge the *Spine Matrix* Ω from five interlocked seals.
(I) The Staircase Seal. Let

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let u be the $(1,2)$ entry of J^5.
(II) The Mirror Seal. Using this u, define the 4×4 matrix

$$L(u) = \begin{pmatrix} 2u & 2 & u & u \\ u & 1 & 1 & 1 \\ 2 & 2 & u^2 & u \\ 1 & 1 & u & 1 \end{pmatrix}.$$

(III) The Choir Seal. Set $n = u - 2$ and $\alpha = \dfrac{\pi}{n+1}$. Define the $n \times n$ matrix

$$A_{m,j} = \sin(mj\alpha) \qquad (1 \le m, j \le n),$$

and the $2n \times 2n$ matrix

$$S = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

(IV) The Black–White Wall. Let $m = 2u + 1$ and define the $m \times m$ matrix W by

$$W_{i,j} = \begin{cases} 1, & i \le j, \\ -1, & i > j. \end{cases}$$

(V) The Imaginary Bolt. Define the $2m \times 2m$ matrix

$$C = \begin{pmatrix} W & -W \\ W & W \end{pmatrix}.$$

The Spine. Let

$$\Omega = \operatorname{diag}\big(L(u),\, S,\, C\big).$$

Vault Format. There is a unique integer $E \ge 0$ and a unique odd integer Q such that

$$\det(\Omega) = Q \cdot 2^E.$$

Vault Directive

Output E, the height of the verdict.

Solution

Ω is block diagonal, so

$$\det(\Omega) = \det(L(u))\det(S)\det(C),$$

and $E = \nu_2(\det L) + \nu_2(\det S) + \nu_2(\det C)$.

Staircase. $J = I + N$ with $N^4 = 0$. Then $J^5 = \sum_{k=0}^{3} \binom{5}{k} N^k$, so the $(1,2)$ entry is $\binom{5}{1} = 5$. Thus $u = 5$, hence $n = u - 2 = 3$, $m = 2u + 1 = 11$.

Mirror. The built-in identity $\det(L(u)) = (u-1)^2(u-2)^2$ gives

$$\det(L(5)) = 4^2 \cdot 3^2 = 9 \cdot 2^4 \;\Rightarrow\; \nu_2 = 4.$$

Choir. For $n = 3$, $\alpha = \pi/4$. Standard sine orthogonality yields $A^2 = \frac{n+1}{2} I = 2I$, so $\det(A)^2 = \det(2I_3) = 2^3$. Also $\det\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right) = (-1)^n \det(A)^2$, hence $\det(S) = -2^3$ and $\nu_2 = 3$.

Bolt. For the "wall" matrix W of size $m = 11$, one has $\det(W) = 2^{m-1} = 2^{10}$ (row-add then triangularize). Now

$$C = \begin{pmatrix} W & -W \\ W & W \end{pmatrix} \xrightarrow{\text{row } 2 \leftarrow 2+1} \begin{pmatrix} W & -W \\ 2W & 0 \end{pmatrix},$$

so factoring 2 from the bottom m rows gives $\det(C) = 2^m \det\left(\begin{smallmatrix} W & -W \\ W & 0 \end{smallmatrix}\right)$. Since W is invertible, the Schur complement gives $\det\left(\begin{smallmatrix} W & -W \\ W & 0 \end{smallmatrix}\right) = \det(W)^2$. Thus

$$\det(C) = 2^{11} \cdot (2^{10})^2 = 2^{31} \;\Rightarrow\; \nu_2 = 31.$$

Therefore $E = 4 + 3 + 31 = 38$.

Answer

38

13

Five Lemmas from the Combinatorics Vault

Combinatorics • Design Theory • Lattice Geometry

This is a gauntlet of five advanced combinatorial claims. You must verify equality cases for antichains, properties of symmetric block designs, and distribution uniformity for lattice points.

Lore

A brass plate hangs from a chain in the Combinatorics Vault, scratched with five "lemmas." Some are real, copied from old seminar notes; some are traps that survive only until someone tries one careful example.
A rim inscription fixes the ground rules the way the Vault likes them fixed:

Count lattice points only in $\mathbb{Z}^d$*. "Residue class mod* p*" means* $r + p\mathbb{Z}^d$ *with* $r \in \{0, 1, \dots, p-1\}^d$.

The clerk below the plate does not argue. The clerk only asks for a mark on each line: working key or broken one.

Puzzle

For each of the following five statements, assign a truth value: assign 1 if the statement is true and 0 if it is false. Express your final answer as a 1×5 matrix

$$T = [t_1, t_2, t_3, t_4, t_5].$$

1. **Antichain equality-case lemma.** Let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain (no set contains another). Define the Lubell–Yamamoto–Meshalkin weight
$$W(\mathcal{A}) = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}.$$
Claim: If $W(\mathcal{A}) = 1$, then $\mathcal{A}$ is exactly the full k-th level $\binom{[n]}{k}$ for some k.
2. **Symmetric incidence matrix lemma.** Let M be an $n \times n$ matrix with entries in $\{0, 1\}$. Assume each column has exactly r ones, and for any two distinct columns, the number of rows in which they both have a 1 is exactly λ. *Claim:* Every row also has exactly r ones, and necessarily
$$r(r-1) = \lambda(n-1).$$
3. **Derangement cycle alternation lemma.** For $n \geq 2$, let $D_n \subset S_n$ be the set of derangements (permutations with no fixed points). For $\pi \in S_n$, let $c(\pi)$ be the number of cycles of π in its disjoint cycle decomposition (including fixed points). *Claim:*
$$\sum_{\pi \in D_n} (-1)^{c(\pi)} = -(n-1).$$
4. **Residue class uniformity lemma.** Let $P \subset \mathbb{R}^d$ be a centrally symmetric convex polytope (i.e. $x \in P \Rightarrow -x \in P$). Let p be prime, and let
$$|P| := |P \cap \mathbb{Z}^d|$$
denote the number of lattice points in P (including boundary points). If
$$|P| = m\, p^d,$$
Claim: for every residue vector $r \in (\mathbb{Z}/p\mathbb{Z})^d$ we have
$$\left|P \cap (r + p\mathbb{Z}^d)\right| = m.$$
5. **Alternating binomial dominance lemma.** Let $a_0, \dots, a_n$ be nonnegative real numbers. If
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k = 0,$$
Claim: there exists a *unique* index j such that
$$\binom{n}{j} a_j > \binom{n}{k} a_k \qquad \text{for all } k \neq j.$$

Vault Directive

Output the 5-bit string $t_1 t_2 t_3 t_4 t_5$.

Solution

1. **True.** LYM gives $W(\mathcal{A}) \le 1$ for any antichain, and equality forces $\mathcal{A} = \binom{[n]}{k}$ for some k.
2. **True.** Column sums $= r$ and constant pairwise intersections λ imply $M^{\mathsf{T}}M = (r-\lambda)I + \lambda J$. Hence $M^{\mathsf{T}}M\mathbf{1} = (r + \lambda(n-1))\mathbf{1}$, so the row-sum vector is constant; counting ones gives every row sum $= r$. Double-count ordered triples (row, $i \ne j$) with a 1 in both columns: by columns: $n(n-1)\lambda$; by rows: $nr(r-1)$. Thus $r(r-1) = \lambda(n-1)$.
3. **True.** Using $\sum_{\pi\in S_m} x^{c(\pi)} = x(x+1)\cdots(x+m-1)$, we get $\sum_{\pi\in S_m}(-1)^{c(\pi)} = 0$ for $m \ge 2$, while $S_0 = 1, S_1 = -1$. Inclusion–exclusion over fixed points yields
$$\sum_{\pi\in D_n}(-1)^{c(\pi)} = \sum_{j=0}^{n}\binom{n}{j}S_{n-j} = S_0 + nS_1 = -(n-1).$$
4. **False.** For $d = 2, p = 3$, take the centrally symmetric lattice parallelogram with vertices $(-3,-6), (-5,-3), (3,6), (5,3)$. It has $|P| = 45 = 5\cdot 3^2$ (e.g. Pick's theorem), but the class $(1,1)$ mod 3 contributes only 3 lattice points in P, not 5.
5. **False.** Let $n = 3$ and $a_0 = a_1 = a_2 = a_3 = 1$. Then $\sum_{k=0}^{3}(-1)^k\binom{3}{k}a_k = 1 - 3 + 3 - 1 = 0$, but $\binom{3}{1}a_1 = \binom{3}{2}a_2$, so there is no unique maximizing j.

Therefore $T = [1, 1, 1, 0, 0]$.

Answer

11100

14

Ω ↑ ⊕

∃! ↑ ⊕ ⊢

$\mathcal{L}_{sym} \models \mathbb{B} \wedge (\uparrow, \oplus) \Rightarrow \exists! \nu$

Lore

$\tau = 22{:}20,\ \Phi = \varnothing.$ $S = \{1,2,3,4\}.$ $\exists i \in S\big(L(i) \wedge \forall j \in S\,(j \neq i \to \neg L(j))\big),$ $\exists i \in S\big(G(i) \wedge \forall j \in S\,(j \neq i \to \neg G(j))\big).$
$\forall i \in S\,\forall k \in \{1,2,3\}:\ (T(i) \to \sigma_{i,k}) \wedge (L(i) \to \neg\sigma_{i,k}).$ $\kappa, \lambda, \Delta.$

Puzzle

$$S = \{1,2,3,4\}.$$

$$T(i) := L(i) \uparrow L(i). \qquad p \uparrow q := \neg(p \wedge q). \qquad p \oplus q := (p \vee q) \wedge \neg(p \wedge q).$$

$$(\exists i \in S\, L(i)) \wedge (\forall a \in S\,\forall b \in S:\ (a \neq b) \to \neg(L(a) \wedge L(b))).$$
$$(\exists i \in S\, G(i)) \wedge (\forall a \in S\,\forall b \in S:\ (a \neq b) \to \neg(G(a) \wedge G(b))).$$
$$\forall i \in S\,\forall k \in \{1,2,3\}:\ (T(i) \to \sigma_{i,k}) \wedge (L(i) \to \neg\sigma_{i,k}).$$

$$\begin{aligned}
\sigma_{1,1} &:= (L(3) \oplus G(4)) \uparrow (L(3) \oplus G(4)) \\
\sigma_{1,2} &:= L(2) \uparrow G(2) \\
\sigma_{1,3} &:= L(4) \uparrow (G(1) \uparrow G(1)) \\
\sigma_{2,1} &:= L(1) \oplus G(4) \\
\sigma_{2,2} &:= G(1) \uparrow (L(3) \uparrow L(3)) \\
\sigma_{2,3} &:= L(3) \uparrow G(2) \\
\sigma_{3,1} &:= G(4) \uparrow G(4) \\
\sigma_{3,2} &:= L(1) \oplus L(2) \\
\sigma_{3,3} &:= G(4) \oplus L(3) \\
\sigma_{4,1} &:= L(3) \uparrow G(2) \\
\sigma_{4,2} &:= (G(4) \oplus (G(1) \uparrow G(1))) \uparrow (G(4) \oplus (G(1) \uparrow G(1))) \\
\sigma_{4,3} &:= L(3) \uparrow G(1)
\end{aligned}$$

Vault Directive

$i^\star := \iota i \in S\ G(i).\ i^\star?.$

Solution

$$\forall a \in S \forall b \in S : (a \neq b) \to \neg(L(a) \land L(b)). \qquad \forall a \in S \forall b \in S : (a \neq b) \to \neg(G(a) \land G(b)).$$

$$\boxed{L(1)} \Rightarrow \neg\sigma_{1,2} \Rightarrow \neg(L(2) \uparrow G(2)) \Rightarrow (L(2) \land G(2)) \Rightarrow (L(1) \land L(2)) \Rightarrow \bot.$$

$$\boxed{L(2)} \Rightarrow \neg\sigma_{2,2} \Rightarrow \neg\big(G(1) \uparrow (L(3) \uparrow L(3))\big) \Rightarrow \big(G(1) \land (L(3) \uparrow L(3))\big).$$

$$L(2) \Rightarrow T(1) \land T(3) \Rightarrow \sigma_{1,1} \land \sigma_{3,3}.$$

$$\sigma_{1,1} \land (L(3) \uparrow L(3)) \Rightarrow (G(4) \uparrow G(4)), \qquad \sigma_{3,3} \land (L(3) \uparrow L(3)) \Rightarrow G(4),$$

$$\Rightarrow \bot.$$

$$\boxed{L(4)} \Rightarrow \neg\sigma_{4,1} \Rightarrow \neg(L(3) \uparrow G(2)) \Rightarrow (L(3) \land G(2)) \Rightarrow (L(3) \land L(4)) \Rightarrow \bot.$$

$$\Rightarrow L(3).$$

$$L(3) \Rightarrow \neg\sigma_{3,1} \Rightarrow \neg(G(4) \uparrow G(4)) \Rightarrow G(4).$$

$$\forall a \in S \forall b \in S : (a \neq b) \to \neg(G(a) \land G(b)) \Rightarrow i^\star = 4.$$

Answer

4

15

The Oath of Two Echoes

Algebra • Hidden Symmetries • Unique Integer

This puzzle relies on the relationship between the coefficients of a polynomial and the symmetric sums of its roots. You must construct a linear system to solve for the unknown products without finding the roots explicitly.

Lore

The Algebra Vault sells "bonds" on roots the way a bazaar sells grain: by weight, not by story.
A quartic $f(x) \in \mathbb{Z}[x]$ is posted on the wall, and four roots r_1, r_2, r_3, r_4 are recorded—two rational, two irrational, all distinct.
The clerk will not sell the roots themselves. The clerk sells only two products: $B_1 = r_1 r_2$ and $B_2 = r_3 r_4$.
The price is old-fashioned: if $B = p/q$ in lowest terms, you pay $p + q$ coins.

Puzzle

The Echo Gauge is governed by the plate

$$169x^4 - 39x^3 + 428x^2 + 89x + 203.$$

It has exactly four echoes r_1, r_2, r_3, r_4 (not necessarily real): these are the four values of x that make the gauge's hum vanish, i.e. the four roots of the equation

$$169x^4 - 39x^3 + 428x^2 + 89x + 203 = 0.$$

You are told that, after relabeling if needed,

$$r_1 + r_2 = \frac{8}{13} \qquad \text{and} \qquad r_3 + r_4 \neq r_1 + r_2.$$

Define the two **bond-prices**

$$B_1 := r_1 r_2, \qquad B_2 := r_3 r_4.$$

The Vault mints coins by the rule:

> Write each bond-price B_i as a reduced fraction $\frac{p_i}{q_i}$ with $\gcd(p_i, q_i) = 1$ and $q_i > 0$. The bond costs $p_i + q_i$ coins.

Vault Directive

Output the total number of coins required to buy both bonds.

Solution

Let the elementary symmetric sums be

$$s_1 = r_1 + r_2 + r_3 + r_4 = -\frac{b}{a}, \quad s_2 = \sum_{i<j} r_i r_j = \frac{c}{a}, \quad s_3 = \sum_{i<j<k} r_i r_j r_k = -\frac{d}{a}.$$

Given $r_1 + r_2 = u = \frac{8}{13}$, and since $s_1 = -\frac{b}{a} = \frac{39}{169} = \frac{3}{13}$, we have $r_3 + r_4 = v = s_1 - u = -\frac{5}{13}$. In particular $u \neq v$.
Set $x = r_1 r_2 = B_1$, $y = r_3 r_4 = B_2$. Then

$$s_2 = (r_1 r_2 + r_3 r_4) + (r_1 + r_2)(r_3 + r_4) = x + y + uv,$$

so $x + y = s_2 - uv$ is rational. Also

$$s_3 = r_1 r_2 (r_3 + r_4) + r_3 r_4 (r_1 + r_2) = vx + uy,$$

so (x, y) solves the 2×2 linear system

$$\begin{cases} x + y = s_2 - uv, \\ vx + uy = s_3, \end{cases}$$

whose determinant $u - v \neq 0$. Hence $x, y \in \mathbb{Q}$ uniquely.
Now plug in the given $s_2 = \frac{428}{169}$, $s_3 = -\frac{89}{169}$, $u = \frac{8}{13}$, $v = -\frac{5}{13}$:

$$uv = -\frac{40}{169} \;\Rightarrow\; x + y = \frac{468}{169}.$$

And $vx + uy = s_3$ becomes $-5x + 8y = -\frac{89}{13}$. Solving gives $x = \frac{29}{13}$, $y = \frac{7}{13}$.
Prices: $(29 + 13) + (7 + 13) = 42 + 20 = 62$.

Answer

62

16

The Twin-Wick Corridor

Lore Logic • Counting by Parity • Unique Integer

This problem asks for a ratio, which suggests you should not compute the massive numerator and denominator separately. Instead, look for a structural relationship: given a specific sequence of stations, exactly how many ways can you assign the wicks?

Lore

The Corridor of Wicks is lit in pairs: each station a matched cradle of brass with a single number stamped deep enough to catch soot.
At every station a Sun wick and a Shade wick share that number; a pull flips the state with a soft click, lit to dark and dark to lit.
The steward keeps the corridor quiet, because the sound of toggling carries.
An old line is etched into the iron at eye height:

> *"The Shade must never shine—unless you walk the reckless road."*

Two rites are copied in the ledger. The Vault accepts only their ratio.

Puzzle

There are **9 numbered stations**, labeled $1, 2, \ldots, 9$. At each station i there are exactly two wicks: a **Sun** wick and a **Shade** wick. All wicks start **dark**.
A **pull** chooses a station i and then chooses *exactly one* of its two wicks; that wick flips state (dark $\leftrightarrow$ lit).
A **rite** is a sequence of exactly **21 pulls**. Two rites are distinct iff they differ at some pull index.
At the end of the 21 pulls, the door demands:

- all 9 Sun wicks are **lit**, and
- all 9 Shade wicks are **dark**.

Define:

- N: the number of **Reckless Rites** that satisfy the final demand (Shade wicks may be lit during the rite).
- M: the number of **Oathbound Rites** that satisfy the final demand and in which **after every prefix of the 21 pulls, all 9 Shade wicks are dark**.

Vault Directive

Compute the integer factor $\dfrac{N}{M}$.

Solution

A wick ends **lit** exactly when it is pulled an *odd* number of times, and ends **dark** exactly when it is pulled an *even* number of times.

Oathbound rites never pull Shade.
In an Oathbound rite, each Shade wick is dark after every prefix. Since every Shade wick starts dark, pulling a Shade wick even once would make it lit immediately after that pull, violating the oath. So in an Oathbound rite, *no Shade wick is ever pulled.*
Therefore an Oathbound rite consists of 21 pulls on Sun wicks only. To satisfy the final demand, each Sun wick must be pulled an odd number of times. So M counts exactly the length-21 sequences of Sun-pulls in which each of the 9 Sun wicks is used an odd number of times.

The station-number skeleton.
Given any rite, forget whether each pull was Sun or Shade and record only its station number. This produces a length-21 *skeleton* word in $\{1, 2, \dots, 9\}$.
For a successful Reckless rite, fix a station i and write

$$s_i = \#(\text{Sun pulls at station } i), \qquad h_i = \#(\text{Shade pulls at station } i), \qquad c_i = s_i + h_i.$$

The final demand requires s_i to be odd (Sun ends lit) and h_i to be even (Shade ends dark), hence c_i is odd. So in the skeleton, each symbol i appears an odd number of times.
Conversely, any skeleton in which each $i \in \{1, \dots, 9\}$ appears an odd number of times determines exactly one Oathbound rite: at each occurrence of i, pull the *Sun* wick at station i. This is valid and satisfies the final demand.
Hence the set of possible skeletons is in bijection with the set of Oathbound rites, so the number of skeletons is exactly M.

Expanding one skeleton into Reckless rites.
Fix one skeleton, and let c_i be the number of occurrences of station i in it. Then each c_i is odd (hence $c_i \geq 1$) and $\sum_{i=1}^{9} c_i = 21$.
To obtain a successful Reckless rite from this skeleton, for each station i we must decide, among the c_i occurrences of i, which are *Shade* pulls (the rest are *Sun* pulls). The final demand requires the Shade wick to end dark, so the number of Shade pulls at station i must be even. Thus we must choose an even-sized subset of the c_i occurrences.
Among the 2^{c_i} subsets, exactly half have even size (toggle a distinguished occurrence to pair even and odd subsets), so there are 2^{c_i-1} valid choices for station i.
These choices are independent across stations, so the number of successful Reckless rites with this fixed skeleton is

$$\prod_{i=1}^{9} 2^{c_i-1} = 2^{(c_1+\cdots+c_9)-9} = 2^{21-9} = 2^{12} = 4096.$$

Thus each skeleton contributes exactly 4096 Reckless rites. Since the number of skeletons is M, we have $N = 4096\,M$, and therefore

$$\frac{N}{M} = 4096.$$

Answer

4096

IV
Labyrinth of Forms

Across these corridors lie shapes, games, patterns, and tilings. Geometry meets invariants, strategy meets symmetry. Here, a single path traces the echo of forms and balance.

17

The Mirror Gate of Caldera

The Obsidian Bisectors

This is a coordinate geometry puzzle focusing on lines and angles. You must decode the inscription to find the primary lines, then calculate a sequence of angle bisectors based on specific geometric constraints.

Lore

In Caldera's basalt gate, three seams look as if they were drawn with light and then cooled into stone: two slanting rails and one vertical spine at $x = 1$. Near the rails, someone has scratched the numbers 7 and 13 like a reminder left for the next worker.
A soot-stained slip has been wedged into the mortar at ankle height:

> *At the spine, take the sharp split. At the right side, take the wide. When a choice remains, keep the ray that points where it's told.*

Arin's rule is simple: every "mirror" line demands an internal bisector, and the gate remembers the choices.

Puzzle

In the usual xy–coordinate plane, the Gate's inscription is

$$\big((x-y)^2 - 20(x-y) + 91\big)(x-1) = 0.$$

Direction-angle convention. The *direction angle* of a *ray* is the angle in $[0°, 360°)$ measured counterclockwise from the positive x–axis to that ray. The direction angle of a *line* is defined to be the direction angle of its ray with positive y–component (so it lies in $[0°, 180°)$; a vertical line has direction angle $90°$).
The set of points satisfying the inscription is the union of three lines. Call the vertical line the Spine s. Call the two parallel slanted lines $\ell_{\odot}$ and ℓ_{M} so that $\ell_{\odot}$ is closer to the origin.
Now build mirrors as follows (each mirror is a line through the indicated hinge-point):

- At $\ell_{\odot} \cap s$, place mirror h along the *internal* bisector of the *acute* angle between $\ell_{\odot}$ and s.
- At $\ell_{\mathrm{M}} \cap s$, look in the half-plane $x > 1$ and take the *obtuse* angle formed there by the rays of ℓ_{M} and s; place mirror g along its *internal* bisector.
- Mirror h meets ℓ_{M} at a point away from the Gate. There, place mirror j along the *internal* angle bisector of ℓ_{M} and h *whose ray with positive y–component points into the half-plane $x > 1$*.
- Mirrors j and g intersect; place the final mirror m along the *internal* angle bisector of j and g *whose ray with positive y–component points into the half-plane $x < 1$*.

Let δ be the direction angle of mirror m (in the line sense above).

Vault Directive

Determine δ as a reduced rational multiple of π.

Solution

Factor the inscription using $t = x - y$:

$$(t^2 - 20t + 91)(x - 1) = 0 = (t - 7)(t - 13)(x - 1),$$

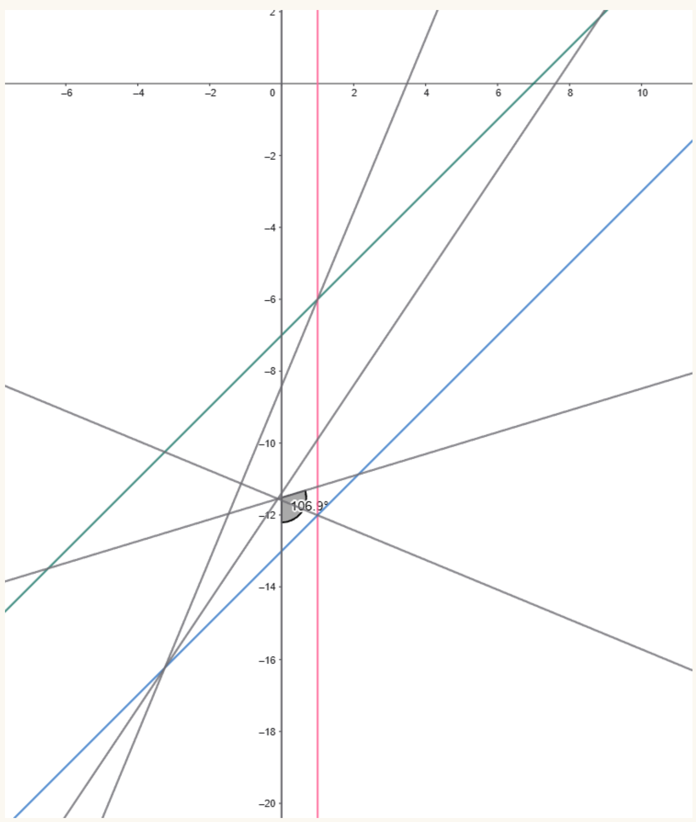
so $\ell_{\odot} : y = x - 7$, $\ell_{\mathrm{M}} : y = x - 13$ (both direction 45°), and $s : x = 1$ (direction 90°).

h: acute bisector of 45° and 90°, so $\theta_h = 67.5° = \frac{135}{2}^{\circ}$.
g: in $x > 1$, the obtuse angle is between the 45° ray of ℓ_{M} and the downward vertical ray 270°, of size 135°. Its bisecting ray has direction $270° + 67.5° = 337.5°$; as a *line* we take the opposite direction with positive y-component: $\theta_g = 337.5° - 180° = 157.5° = \frac{315}{2}^{\circ}$.
j: internal bisectors between 45° and 67.5° are 56.25° and 146.25°; the $x > 1$ rule selects $\theta_j = 56.25° = \frac{225}{4}^{\circ}$.
m: internal bisectors between 56.25° and 157.5° are

$$\frac{56.25° + 157.5°}{2} = 106.875° \quad \text{and} \quad 16.875°;$$

the $x < 1$ rule selects $\delta = 106.875° = \frac{855}{8}^{\circ}$. Thus

$$\delta = \frac{855}{8} \cdot \frac{\pi}{180} = \frac{19\pi}{32}.$$



Answer

$\delta = \dfrac{19\pi}{32}$.

18

The Brass Protractor Rite

Classical Construction • Tangents • Reflections

This is a rigorous construction puzzle. You must follow the straightedge-and-compass instructions precisely to generate a specific geometric figure, then calculate the sum of five resulting angles.

Lore

The archivists recovered no diagram—only a sequence of geometric rites etched into vellum and sealed in brass.
Each segment, arc, and reflection had been chosen not for ornament, but for invocation. The angles were not decoration. They were a signature.
They say Athena devised the ritual herself—not to guard the Vault, but to remind it what precision feels like.
To unlock the seal, one must reconstruct her rite and calculate the sacred total.

Puzzle

All angles $\angle XYZ$ denote the *smaller* angle at vertex Y (between 0° and 180°).
A courtyard designer performs the following straightedge-and-compass construction:

1. Begin with a square labeled clockwise as S_1, T_1, U_1, R_1.
2. On side R_1U_1, construct an equilateral triangle $\triangle R_1V_1U_1$ with apex V_1 chosen *outside* the square.
3. Draw the circle centered at R_1 with radius R_1U_1.
4. Using segment R_1V_1 as a side, construct a regular pentagon $R_1V_1W_1Z_1A_2$ in clockwise order, chosen so that the pentagon lies in the half-plane determined by line R_1V_1 *opposite* U_1.
5. Since A_2 lies on the circle centered at R_1, draw the tangent to that circle at A_2. Let this tangent meet the extension of line Z_1W_1 at B_2.
6. Draw the circle centered at S_1 with radius S_1T_1.
7. Let C_2 be the intersection point of the two circles (centered at R_1 and S_1, equal radii) that lies *outside* the square and in the same open half-plane bounded by line R_1S_1 as Z_1. Draw the tangent to the circle centered at S_1 at C_2. Let this tangent meet line R_1A_2 at D_2.
8. Let the same tangent line from C_2 meet line V_1W_1 at E_2.
9. Let M_2 be the intersection point of the two tangents (the tangent at A_2 to the circle centered at R_1, and the tangent at C_2 to the circle centered at S_1).
10. Reflect point B_2 across the tangent line at C_2 (i.e., across the line C_2D_2), and call the image G_2.
11. Reflect point A_2 about point C_2 (a half-turn about C_2), and call the image H_2.

Vault Directive

Compute

$$\angle Z_1B_2A_2 \;+\; \angle D_2M_2A_2 \;+\; \angle D_2E_2V_1 \;+\; \angle R_1G_2U_1 \;+\; \angle S_1C_2H_2,$$

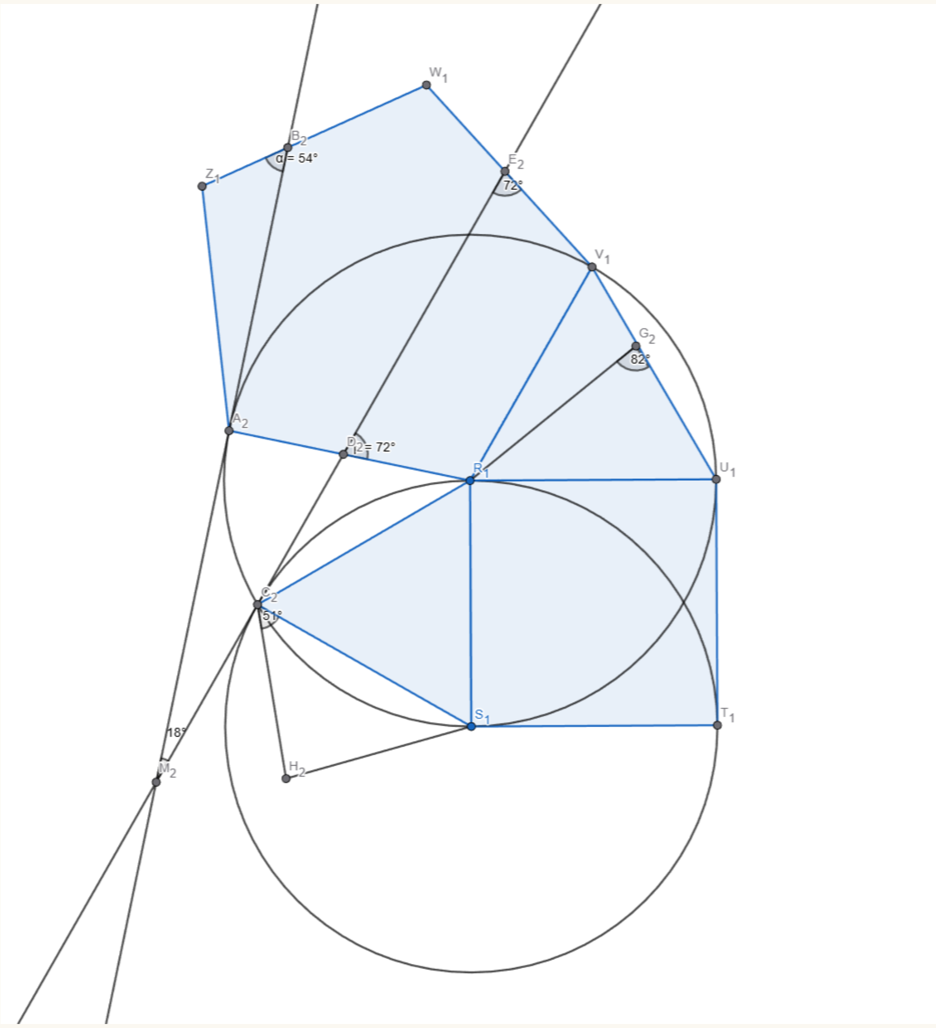
and report the value to the *nearest whole degree*.

Solution

Fix the square side to 1 with $S_1(0,0)$, $T_1(1,0)$, $U_1(1,1)$, $R_1(0,1)$. The equilateral apex is $V_1(\frac{1}{2},\ 1+\frac{\sqrt{3}}{2})$. The circles meet at $C_2(-\frac{\sqrt{3}}{2},\ \frac{1}{2})$; the tangent there is $-\sqrt{3}x+y=2$.
Constructing the regular pentagon $R_1V_1W_1Z_1A_2$ determines the tangent at A_2 and the lines Z_1W_1 and V_1W_1. Solving for the intersections yields B_2, D_2, E_2, M_2. Reflections yield G_2 and H_2. Calculating vector angles gives:

$$\angle Z_1B_2A_2 = 54°, \quad \angle D_2M_2A_2 = 18°, \quad \angle D_2E_2V_1 = 72°, \quad \angle S_1C_2H_2 = 51°,$$

and the irrational angle $\angle R_1G_2U_1 \approx 81.9786°$. The sum is $54+18+72+51+81.9786\cdots = 276.9786\ldots°$, which rounds to **277**.



Answer

277

19

The Three Bead Readings

Geometry • Vault Mechanics • Ratios

This puzzle uses vector geometry and locus definitions. You must convert ”secant ratios” into distance equations to find the locus of the point P, and then use the ”midpoint distances” to calculate the size of that locus.

Lore

In the Astral Annex, diviners read geometry the way accountants read ledgers: by invariants.
A sphere of radius $R = 9$ is suspended on three chains. Four beads W, X, Y, Z are fixed to the glass, and a fifth point P may wander. The diviners do not measure angles directly. They measure *lengths* and *powers*, then declare whether a reading can be made.
Three bead-readings are written in the margin: the squared distances between the midpoints of opposite bead-pairs. The rest, they claim, can be deduced.

Puzzle

The vault's *heart* is the point H that is equally distant from every point of the wall; that common distance is 9.
A lantern is held at a point P strictly inside the vault. For each nail (say W), extend the straight line through W and P until it meets the wall again on the far side; call that second intersection $W_1 \neq W$. Define the *weighing* of nail W to be the ratio

$$\frac{WP}{PW_1},$$

where both lengths are measured along the same straight line $W - P - W_1$. Thus the four weighings are $\frac{WP}{PW_1}, \frac{XP}{PX_1}, \frac{YP}{PY_1}, \frac{ZP}{PZ_1}$.
The brass rule says the sum of the four weighings equals 4:

$$\frac{WP}{PW_1} + \frac{XP}{PX_1} + \frac{YP}{PY_1} + \frac{ZP}{PZ_1} = 4.$$

Separately, the ledger describes the keeper's *pairing ritual*:

- On a night, the keeper connects two disjoint pairs of nails with taut wire (so every nail is used exactly once that night).
- The keeper presses a wax bead to the midpoint of each wire.
- The keeper measures the straight distance between the two beads.
- The keeper repeats until, across all nights, *every pair of nails has been connected exactly once*.

The three recorded bead-distances (in no particular order) are

$$12, \quad 8, \quad 4.$$

The Archive claims that the set of all lantern-positions P obeying the brass rule forms one perfectly round *inner shell*.

Vault Directive

What is the **diameter length** of that inner shell?

Solution

Let H be the sphere's center ($R = 9$). For any bead $A \in \{W, X, Y, Z\}$, the secant through P meets the sphere again at A_1, and

$$PA \cdot PA_1 = R^2 - HP^2 \quad \Rightarrow \quad \frac{PA}{PA_1} = \frac{PA^2}{R^2 - HP^2}.$$

So the Oracle equation $\sum_A \frac{PA}{PA_1} = 4$ is equivalent to

$$\sum_A PA^2 = 4(R^2 - HP^2). \tag{1}$$

Let C be the centroid of $WXYZ$. The identity $\sum_A \overrightarrow{PA} = 4\overrightarrow{PC}$ implies the quadratic decomposition

$$\sum_A PA^2 = 4\,PC^2 + \sum_A CA^2. \tag{2}$$

Apply (2) at $P = H$: since $HA = R$ for all beads,

$$4R^2 = \sum_A HA^2 = 4\,HC^2 + \sum_A CA^2 \;\Rightarrow\; \sum_A CA^2 = 4(R^2 - HC^2). \tag{3}$$

Substitute (2) and (3) into (1):

$$4PC^2 + 4(R^2 - HC^2) = 4(R^2 - HP^2) \implies HP^2 + PC^2 = HC^2.$$

Thus the locus of P is exactly the sphere whose diameter is HC. So the asked diameter is HC.
Let a_W, a_X, a_Y, a_Z be the position vectors of the beads and $c = \frac{a_W + a_X + a_Y + a_Z}{4}$ ($|c| = HC$). Then

$$\sum_{i<j} \|a_i - a_j\|^2 = 4\sum_i \|a_i\|^2 - \Big\|\sum_i a_i\Big\|^2 = 16R^2 - 16HC^2 = 16(R^2 - HC^2). \tag{4}$$

For midpoints M_{WX}, M_{YZ} etc.,

$$\|M_{WX} - M_{YZ}\|^2 = \tfrac{1}{4}\|(W + X) - (Y + Z)\|^2,$$

and summing the three "opposite-midpoint" squares gives

$$d_1^2 + d_2^2 + d_3^2 = \frac{1}{4}\sum_{i<j} \|a_i - a_j\|^2. \tag{5}$$

Given $d_1 = 12, d_2 = 8, d_3 = 4$, we have $d_1^2 + d_2^2 + d_3^2 = 12^2 + 8^2 + 4^2 = 224$, so by (5) and (4):

$$224 = \frac{1}{4} \cdot 16(R^2 - HC^2) \;\Rightarrow\; R^2 - HC^2 = 56.$$

With $R^2 = 81$, we get $HC^2 = 25$, so $HC = 5$. Therefore the locus sphere has radius $HC/2 = \boxed{\frac{5}{2}}$, so its *diameter* is $\boxed{5}$.

Answer

5

20

The Staircase Registry

Adversarial Search • Symmetrization • Recurrence

A three-stage ledger puzzle: you must extract hidden digits from an adversarial “distance” oracle, then force an extremal friendship structure via symmetrization, and finally convert a staircase matching recurrence into a closed form. Expect an information bound, a structural extremal argument, and an exact enumeration that culminates in one large integer.

Lore

A torn folio from the annex shows three headings, written in the same steady hand:
Echo. Feast. Bindings.
Under *Echo* are three probes and three even replies. Under *Feast* is a single line: *"Make the weak quartets as many as possible."*
Under *Bindings* is the old staircase of wardens and seals.
In the margin: *"If any step is guessed, the next one fails."*

Puzzle

I. The Echo-Ruler. A witness secretly chooses an ordered triple (u, v, w) with $0 \le u, v, w \le 9$. A probe is a triple (a, b, c) with $0 \le a, b, c \le 9$, and the reply is

$$R(a, b, c) = |u + v - a - b| + |v + w - b - c| + |w + u - c - a|.$$

Let M be the minimum number of probes needed to guarantee determining (u, v, w) in the worst case.
The folio records that the auditor used M probes and obtained:

$$R(0, 0, 0) = 22, \qquad R(9, 2, 0) = 14, \qquad R(4, 0, 7) = 10.$$

II. The Feast of Friends. Set $N := 100 + 10u + v$. Among N people, some pairs are friends. A *weak quartet* is a set of four people containing *exactly one* friendship pair. Let $Q(N)$ be the maximum possible number of weak quartets over all friendship graphs on N vertices. Let k be the number of connected components (cliques) in an extremal configuration attaining $Q(N)$.

III. The Staircase Registry. There are 15 wardens $W_1, \dots, W_{15}$ and 29 seals $S_1, \dots, S_{29}$. A *binding* of size r is a set of r disjoint Warden–Seal pairs. It is *legal* if W_i may only bind to seals S_j with $j \le 2i - 1$. Let $A(15, r)$ be the number of legal bindings of size r. Define

$$r := M + k + w, \qquad V := A(15, r) - Q(N).$$

Vault Directive

Compute V.

Solution

I (Echo). Each reply is even and lies in $[0, 54]$, so there are at most 28 possible replies; hence two probes give $\leq 28^2 = 784 < 10^3$ outcomes, so $M \geq 3$. The folio lists M probes and shows three, hence $M = 3$.
From $R(0,0,0) = 22$ we get $u + v + w = 11$. For $(9, 2, 0)$,

$$14 = R(9,2,0) = |u+v-11| + |v+w-2| + |u+w-9| = w + (9 - u) + |u + w - 9|,$$

so $14 = 2w$ if $u + w \geq 9$, and $14 = 18 - 2u$ if $u + w < 9$; thus $w = 7$ or $u = 2$. For $(4, 0, 7)$,

$$10 = R(4,0,7) = |u+v-4| + |v+w-7| + |u+w-11| = |7 - w| + |4 - u| + v.$$

If $w = 7$ then $u + v = 4$ and $R = |4 - u| + v \in \{0, 2, 4, 6, 8\}$, contradiction; hence $u = 2$. Then $v + w = 9$ and $10 = |7 - w| + 2 + v = |v - 2| + v + 2$, giving $v = 5$, $w = 4$. So $(u, v, w) = (2, 5, 4)$.

II (Feast). Now $N = 100 + 10u + v = 125$. By the copying/symmetrization inequality $2q(G) \leq q(G_x) + q(G_y)$, an extremal graph is a disjoint union of cliques. With clique sizes $a_1, \ldots, a_n$ ($\sum a_i = N$),

$$q(G) = \sum_{i=1}^{n} \binom{a_i}{2} \sum_{\substack{j<\ell \\ j,\ell \neq i}} a_j a_\ell.$$

For fixed n, $q(G)$ is maximized when the nonzero a_i differ by ≤ 1; for equal size $a = N/n$,

$$q_n = n\binom{a}{2}\binom{n-1}{2}a^2 = \frac{N^3}{4} \cdot \frac{(n-1)(n-2)(N-n)}{n^3}.$$

For $N = 125$, the factor $f(n) = \frac{(n-1)(n-2)(125-n)}{n^3}$ satisfies $f(4) = \frac{726}{64} < f(5) = \frac{1440}{125}$ and $f(6) = \frac{2380}{216} < f(5)$, so the maximum is at $n = 5$. Since $125 = 5 \cdot 25$, an extremal configuration has $k = 5$ cliques of size 25, and

$$Q(125) = 5\binom{25}{2}\binom{4}{2}25^2 = 5 \cdot 300 \cdot 6 \cdot 625 = 5{,}625{,}000.$$

III (Registry). $r = M + k + w = 3 + 5 + 4 = 12$. The staircase recurrence gives $A(n, r) = (2n - r)A(n - 1, r - 1) + A(n - 1, r)$, hence $A(n, r) = \binom{n}{r}^2 r!$. Thus

$$A(15, 12) = \binom{15}{3}^2 12! = 455^2 \cdot 12! = 99{,}165{,}306{,}240{,}000,$$

so

$$V = A(15, 12) - Q(125) = 99{,}165{,}306{,}240{,}000 - 5{,}625{,}000 = 99{,}165{,}300{,}615{,}000.$$

Answer

99165300615000

21

The Nine Lenses

Visual Game • Pairing Invariant • Unique Name

This is a minimax grid game. You must determine the optimal score one player can force against a perfect opponent by identifying a geometric invariant that limits the maximum sum.

Lore

A scryglass table sits in the Archive Casino: five panes by five, each pane a dark socket in brass.
Two keepers take turns filling the sockets: one can write only *Lumen* and the other can stamp only *Null*.
When the last socket is filled, the glass does not read the whole surface. It reads only its nine viewing lenses (the 3×3 windows), then speaks the brightest total.
A brass ring translates totals into names.
Speak the name the table speaks.

Puzzle

On a 5×5 grid, two players alternately fill empty cells. The first player always writes 1, the second player always writes 0. Exactly one cell is filled per turn, until all 25 cells are filled.
There are exactly nine 3×3 *lenses* (the 3×3 subgrids with consecutive rows and columns). For each lens, compute the sum of the nine entries in it. Let A be the *maximum* of these nine sums.

The ring of names is:

A	Name on the Ring
0	Athena
1	Quill
2	Kestrel
3	Sable
4	Raven
5	Aegis
6	Aletheia
7	Irin
8	Axiom
9	Cyrene

The first player wants to make A as large as possible; the second player wants to make A as small as possible.

Vault Directive

What value of A can the first player guarantee, and what *name* on the ring corresponds to it?

Solution

We show the second player can keep $A \le 6$, and the first player can force $A \ge 6$. Hence $A = 6$ and the accepted name is **Aletheia**.

Step 1: The second player can enforce $A \le 6$ by pairing.
Pair each cell in row 1 with the cell directly below it in row 2 (same column), and pair each cell in row 3 with the cell directly below it in row 4. (Row 5 is unpaired.) So there are 10 disjoint pairs.

Strategy. Whenever the first player writes 1 in a cell that belongs to one of these pairs, the second player immediately writes 0 in the other cell of the same pair (if it is still empty). Otherwise the second player plays anywhere.
Then each paired two-cell block contains at most one 1, hence at least one 0.
Now take any 3×3 lens. It uses three consecutive rows, so it contains either both rows 1 and 2, or both rows 3 and 4. In that two-row block, the lens contains three complete pairs (one in each of its three columns), so the lens contains at least three 0's. Therefore every lens has sum at most $9 - 3 = 6$, so $A \le 6$.

Step 2: The first player can force $A \ge 6$.
Label rows a, b, c, d, e (top to bottom) and columns $1, 2, 3, 4, 5$ (left to right).

Move 1. First plays 1 at the center cell $c3$.
Let the second player's first 0 be at some cell $q \ne c3$. Consider the four "bands": top three rows $\{a, b, c\}$, bottom three rows $\{c, d, e\}$, left three columns $\{1, 2, 3\}$, and right three columns $\{3, 4, 5\}$. Their intersection is exactly $\{c3\}$, so q lies outside at least one band. First chooses such a band and plays the center-adjacent cell inside it. By rotation/reflection symmetry, we may assume the chosen band is the bottom three rows $\{c, d, e\}$, so First plays 1 at $d3$ on the second move by First, and the first 0 lies outside rows $\{c, d, e\}$.
Now consider the second player's second move.

Case 1: Second does *not* play at $e3$.
Then First plays 1 at $e3$, so First owns the vertical triple $c3, d3, e3$. Let S be the 12 cells in rows $\{c, d, e\}$ but not in column 3. At this moment, the first 0 is outside rows $\{c, d, e\}$ and the second 0 is not at $e3$, so at most one 0 lies in S; hence at least 11 cells of S are still empty.
From now on, whenever any cell of S is empty, First plays in an empty cell of S. Even if Second also plays in S whenever possible, First claims at least $\lfloor 11/2 \rfloor = 5$ cells of S. Split S into $S_L = \{c, d, e\} \times \{1, 2\}$ and $S_R = \{c, d, e\} \times \{4, 5\}$, each of size 6. By pigeonhole, one of S_L, S_R contains at least 3 of First's ones. Together with the triple in column 3, this gives a 3×3 lens with at least 6 ones. So $A \ge 6$ in Case 1.

Case 2: Second *does* play at $e3$ (placing a 0 there).
Then First plays 1 at $c2$. Define the two y-cells to be $d2$ and $e2$, and define the six z-cells to be

$$c1, d1, e1, \quad c4, d4, e4.$$

All eight of these cells lie in rows $\{c, d, e\}$ and are distinct from $c2, c3, d3, e3$, so all eight are empty at this moment.
First now plays by priorities:

- If Second ever plays in one y-cell, then on the next move First plays in the other y-cell.
- Otherwise, whenever any z-cell is empty, First plays in an empty z-cell.
- If no z-cell is empty and both y-cells are still empty, then First plays in a y-cell.

This guarantees that First gets at least one y-cell. Also, among the six z-cells, Second can claim at most three, so First claims at least three z-cells. Since three z-cells lie in column 1 and three in column 4, having at least three z's forces First to have at least two z's in the same one of these columns.
If First has at least two z's in column 1, then the bottom-left lens $\{c, d, e\} \times \{1, 2, 3\}$ contains $c2, c3, d3$ plus those two column-1 ones plus one y-cell in column 2, totaling at least 6 ones. If instead First has at least two z's in column 4, then the lens $\{c, d, e\} \times \{2, 3, 4\}$ has the same conclusion. So $A \ge 6$ in Case 2 as well.

Combining both cases, First can force $A \ge 6$. Together with Step 1, this gives $A = 6$.
On the ring, $A = 6$ corresponds to **Aletheia**.

Answer

Aletheia

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The Talon Plates

Tiling • Local Forcing • One Vault Code

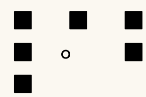
Do not be deceived by the area. Just because the numbers fit the ledger does not mean the plates fit the floor. You must find the 'atom' of the tiling to separate the true chambers from the impossible ones.

Lore

The Archive Casino keeps a repair corridor: a brass runway scored into unit squares.
The steward slides a tray of identical talon plates—six unit-squares fused into a crooked claw. At the bend of every claw is a pocket: a deliberate void the metal leans around but does not cover.
Some chambers accept the claws cleanly; others reject them no matter how patiently the pieces are turned. The copper strip below does not ask *how*; it asks only which chambers can be sealed.

Puzzle

A **talon plate** is the fixed six-square shape below (filled squares are covered; the ∘ is the *pocket square* and is *not* covered):



A talon may be translated, rotated, or reflected.
A rectangle of size $m \times n$ is called **sealable** if its mn unit squares can be tiled by talon plates with no overlaps and no gaps.

The copper strip lists the following chambers:

Rectangle	Rune
14×18	11
8×15	17
10×30	19
12×7	29
6×20	37
22×18	31
16×9	41
12×11	43
18×18	23

Vault Directive

Sum the runes of the sealable chambers.

Solution

We give two necessary conditions (local forcing and a mod-4 obstruction), then tile exactly the chambers that pass.

Linked-pair involution $\Rightarrow 12 \mid mn$.

Fix a tiling of an $m \times n$ rectangle. For any plate H, let $P(H)$ be its *pocket square*: the unique unit square adjacent to three covered squares of H. Let $M(H)$ be the *mouth square* of that pocket: the unique side-adjacent neighbor of $P(H)$ not covered by H.

Because the rectangle is perfectly tiled, $P(H)$ is covered by exactly one other plate; call it $f(H)$. (So $f(H) \neq H$.)

Local forcing: If $K = f(H)$ covers $P(H)$, then K must also cover $M(H)$ (it is the only available neighbor of $P(H)$ not already blocked by H). Checking the finitely many placements of a talon that cover two adjacent squares $P(H)$ and $M(H)$ without overlapping H, one finds that the only placements extendable to a full tiling are exactly those in which H covers the pocket square $P(K)$. Equivalently, $f(f(H)) = H$.

Thus the plates partition into *linked pairs* $\{H, f(H)\}$. Each linked pair covers 12 squares, so mn must be a multiple of 12.

If $12 \mid mn$ but $4 \nmid m$ and $4 \nmid n$, then the rectangle is not sealable.

Assume $12 \mid mn$ and $4 \nmid m, 4 \nmid n$. Then $m \equiv n \equiv 2 \pmod 4$, so the number of linked pairs is

$$\frac{mn}{12},$$

which is odd.

A direct finite check shows that, up to rotation/reflection, the union of any linked pair is one of the following four 12-square shapes:

Now color the rectangle by making every 4th *column* black (columns $4, 8, 12, \ldots$). Since $n \equiv 2 \pmod 4$, the number of black columns is $\lfloor n/4 \rfloor$, so the total number of black squares is

$$m \cdot \lfloor n/4 \rfloor,$$

and since m is even, this number is even.

In a linked pair of type (I) or (III), the shape spans 4 consecutive columns and has exactly 3 squares in each of those columns, so it covers exactly 3 squares in the unique black column among them—an *odd* number. In a linked pair of type (II) or (IV), every column contains an even number of squares, so it covers an *even* number of black squares.

Therefore the number of linked pairs of type (I)/(III) must be even. But the total number of linked pairs is odd, so the number of linked pairs of type (II)/(IV) is odd.

Now repeat the argument with every 4th *row* colored black. The total number of black squares is $n \cdot \lfloor m/4 \rfloor$, again even. Pairs of type (II)/(IV) span 4 consecutive rows and have 3 squares in each row, so each such pair covers an odd number of black squares, while pairs of type (I)/(III) cover an even number. Hence the number of type (II)/(IV) pairs must be even—contradiction.

So if $12 \mid mn$, at least one of m, n must be divisible by 4.

Constructions.

From type (I) above, two talons tile a 3×4 rectangle. Therefore any rectangle whose side lengths are multiples of 3 and 4 is sealable by partitioning it into 3×4 blocks.

Also, since 12 is a multiple of 3 and 4, any $12 \times t$ rectangle with t a sum of 3's and 4's is sealable by splitting into 12×3 and 12×4 strips. In particular:

$$12 \times 7 = (12 \times 3) + (12 \times 4), \qquad 12 \times 11 = (12 \times 3) + (12 \times 4) + (12 \times 4).$$

Evaluate the nine chambers.

All nine rectangles have area divisible by 12. By Step 2, any rectangle with both side lengths $\equiv 2 \pmod 4$ is not sealable. Thus

$$14 \times 18,\ 10 \times 30,\ 22 \times 18,\ 18 \times 18 \quad \text{are not sealable.}$$

The remaining five are sealable by Step 3:

$$8 \times 15,\ 12 \times 7,\ 6 \times 20,\ 16 \times 9 \text{ (multiples of 3 and 4)}, \quad \text{and} \quad 12 \times 11.$$

So the required rune-sum is

$$17 + 29 + 37 + 41 + 43 = 167.$$

Answer

167

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The Mirror Lemma Key

Geometry • Lemma Chain • Unique Answer

This puzzle connects geometry to graph theory. By translating the 'mirror-true' property into integer parity, you can bound the maximum number of specific triangles using the dual tree of the triangulation.

Lore

On a high hook in the Vault hangs a ring of glass like a crown, its rim cut into 2026 facets that catch the lamp in cold, steady flashes.
A pilgrim named Irin is given charcoal and a straightedge and told to cut the ring into triangles, nothing more.
When a triangle settles the glass in just the right way, the crown answers with a clean tone that seems to come from inside the material.
Each tone advances the Mirror Lemma Key by a single click.
The ring does not care how elegant the drawing is—only how many true tones can be forced.

Puzzle

Let Ω be a regular 2026-gon.
A segment joining two vertices of Ω (either a side or a diagonal) is called **mirror-true** if its endpoints split the boundary of Ω into two arcs, each consisting of an *odd* number of sides of Ω.
Irin draws 2023 diagonals of Ω, no two intersecting in the interior, thereby dissecting Ω into 2024 triangles.
A triangle in the dissection is called a **true-tone** triangle if:

- it is isosceles, and
- exactly two of its sides are mirror-true.

Vault Directive

What is the **maximum possible** number of true-tone triangles in such a dissection?

Solution

Label vertices $1, 2, \ldots, 2026$ cyclically, $N = 2026$. For i, j, let $s(i, j) = \min(|i - j|, N - |i - j|)$. The chord ij splits the boundary into arcs of lengths $s(i, j)$ and $N - s(i, j)$, which have the same parity since N is even. So ij is mirror-true iff $s(i, j)$ is odd iff $|i - j|$ is odd, i.e., iff i, j have opposite parity. In particular, every polygon side is mirror-true.

In any triangle, either all three vertices share a parity (then 0 mirror-true sides) or exactly one differs (then exactly the two incident sides are mirror-true). Hence every triangle has 0 or 2 mirror-true sides.

In a regular N-gon, chord length depends only on $s(i, j)$, so equal sides have step sizes of the same parity. Therefore in a true-tone triangle (two mirror-true sides, one non-true side), the equal sides must be the two mirror-true sides; the base is the unique non-true side, hence a diagonal (not a polygon side).

Let T be the dual graph of the triangulation (a tree on 2024 triangles, 2023 edges). Let k be the number of mirror-true diagonals among the drawn diagonals, and delete from T the k edges corresponding to those diagonals, obtaining a forest F with $k + 1$ components. A triangle with 0 mirror-true sides contains no polygon side (all polygon sides are mirror-true), hence all three of its sides are diagonals, all non-true; so it has degree 3 in F. A triangle with 2 mirror-true sides has exactly one non-true side, necessarily a diagonal, so has degree 1 in F. Thus every component of F is a $\{1, 3\}$-degree tree. If such a tree has n vertices and L leaves, then $L + 3(n - L) = 2(n - 1) \Rightarrow L = \frac{n+2}{2}$. Summing over components ($\sum n = 2024$) gives total leaves

$$\sum \frac{n_i + 2}{2} = \frac{2024 + 2(k + 1)}{2} = 1013 + k,$$

so there are $1013 + k$ triangles with 2 mirror-true sides.

A mirror-true diagonal XY cannot border two true-tone triangles. Let $s = s(X, Y)$, which is odd. In a true-tone triangle, the mirror-true sides are the equal sides, so XY must be an equal side and the apex is X or Y. If the apex is X, then the third vertex must be the other vertex (if any) at step size s from X; if the apex is Y, it must be the other vertex (if any) at step size s from Y. In both cases that third vertex lies on the same boundary arc between X and Y, hence on the same side of chord XY. Since the dissection has two triangles adjacent to XY on opposite sides, at most one is true-tone. Hence at least k of the $1013 + k$ leaves are not true-tone, so

$$\#(\text{true-tone}) \leq (1013 + k) - k = 1013.$$

Construction: draw the 1013 noncrossing diagonals $(1, 3), (3, 5), \ldots, (2025, 1)$, carving off the 1013 ear triangles $(1, 2, 3), (3, 4, 5), \ldots, (2025, 2026, 1)$. Each ear is isosceles with exactly two mirror-true sides (the polygon sides). The remaining region uses only odd vertices, so has 0 mirror-true sides in every triangle. Thus 1013 is achievable.
Therefore the maximum is $\boxed{1013}$.

Answer

1013

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The Balancer's Verdict

Geometry • Decision Chain • Five-Letter Code

A convex quadrilateral is cut by rails through its diagonal intersection, producing two corner tiles whose areas must satisfy a square-root inequality. The task is to justify a forced chain of geometric reductions and identify the only consistent choice on each of five tumblers, whose initials reveal the final five-letter code.

W

Lore

The Balancer's Seal is not a lock so much as a verdict.
A quadrilateral is scratched into slate; then two smaller "corner tiles" are cut out by parallels through the diagonal intersection.
The clerk recites the Balance Law as if it were accounting:

> *The whole is at least the sum of the roots of its corner debts.*

Five tumblers hang beneath the slate, each offering four rune-words. Pick one word per tumbler; take their first letters in order; speak the resulting name.

Puzzle

A convex glass floor has four corners P, Q, R, S in that order. The two glass diagonals meet at a pin $O = PR \cap QS$.
Through O, draw four rails parallel to the sides to define the corner cuts: Define points $X \in PQ$ and $Y \in QR$ such that $OX \parallel QR$ and $OY \parallel PQ$. Similarly, define $V \in SP$ and $W \in RS$ such that $OV \parallel RS$ and $OW \parallel SP$.
This partitions the glass into four tiles: $PXOV$, $QXOY$, $RYOW$, $SVOW$, where $QXOY$ and $SVOW$ are parallelograms by construction.
Let $H = [PQRS]$, $A = [PXOV]$, and $B = [RYOW]$, where $[\cdot]$ denotes area.

An engraving on the key claims the **Balance Law**:

$$\sqrt{H} \geq \sqrt{A} + \sqrt{B}.$$

The Balancer Key's five tumblers are labeled I–V. Each tumbler offers four rune-words; exactly one choice per tumbler can be *justified* from the geometry in a way that makes the Balance Law unavoidable in every valid configuration. Choose the correct rune-word on each tumbler to form a 5-letter name.

Tumbler I (What reshaping is allowed that preserves rails and rescales every area uniformly?).

Affine | **Twist** | **Crumple** | **Break**

Tumbler II (What may be imposed after such reshaping to simplify the geometry of the diagonals?).

X-orthogonalize | **Parallelogram** | **Encircle** | **Equalize**

Tumbler III (What hidden alignment appears for the two corner-tiles?).

Induce | **Round** | **Jagged** | **Split**

Tumbler IV (Which area recipe is correct for a *rifted* 4-corner pane?).

Orthocross | **Heron-root** | **Beam × Height** | **Diagonal-song**

Here the recipes mean:

- **Orthocross:** for a quadrilateral whose diagonals are perpendicular, with diagonal lengths d_1, d_2,

$$[\,] = \frac{d_1 d_2}{2}.$$

- **Heron-root:** the triangle formula $\sqrt{p(p-a)(p-b)(p-c)}$.
- **Beam×Height:** base times altitude over 2 (triangle-only).
- **Diagonal-song:** $[\,] = d_1 d_2$ (a tempting song, but missing the factor 1/2).

Tumbler V (Which 1-line identity is the stitching thread?).

Merge | **Average** | **Cancel** | **Swap**

Vault Directive

What 5-letter name appears?

Solution

I. Choose Affine. The construction of X, Y, V, W uses only parallel lines, and any affine map preserves parallelism. All areas scale by a common factor λ, so $\sqrt{H}, \sqrt{A}, \sqrt{B}$ all scale by $\sqrt{\lambda}$; the inequality is invariant.

II. Choose X-orthogonalize. Apply an affine map fixing O that sends the two diagonal directions to perpendicular directions (this is possible since the diagonals are non-parallel); hence we may assume

$$PR \perp QS.$$

III. Choose Induce. In $\triangle PQR$, $O \in PR$ and $OX \parallel QR$, so $\triangle POX \sim \triangle PRQ$, giving $\frac{PX}{PQ} = \frac{PO}{PR}$. In $\triangle PRS$, $OV \parallel RS$ gives $\triangle POV \sim \triangle PRS$, so $\frac{PV}{PS} = \frac{PO}{PR}$. Thus $\frac{PX}{PQ} = \frac{PV}{PS}$, hence $XV \parallel QS$ in $\triangle PQS$. Similarly, $OY \parallel PQ$ and $OW \parallel SP$ yield $\frac{RY}{RQ} = \frac{RO}{PR} = \frac{RW}{RS}$, so $YW \parallel QS$ in $\triangle RQS$. Since $QS \perp PR$ and $PO, RO \subset PR$, we have $PO \perp XV$ and $RO \perp YW$.

IV. Choose Orthocross. With erected diagonals,

$$H = [PQRS] = \frac{PR \cdot QS}{2}.$$

Also $A = [PXOV] = \frac{PO \cdot XV}{2}$ and $B = [RYOW] = \frac{RO \cdot YW}{2}$. From $XV \parallel QS$, $\triangle POV \sim \triangle PRS$ gives $XV = \frac{PO}{PR} QS$; similarly $YW = \frac{RO}{PR} QS$. Hence

$$A = \frac{QS}{2PR} PO^2, \qquad B = \frac{QS}{2PR} RO^2.$$

V. Choose Merge. Therefore

$$\sqrt{A} + \sqrt{B} = \sqrt{\frac{QS}{2PR}}\,(PO + RO) = \sqrt{\frac{QS}{2PR}} \cdot PR = \sqrt{\frac{PR \cdot QS}{2}} = \sqrt{H},$$

so equality holds and the forced rune initials spell `AXIOM`.

Answer

AXIOM

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The Tangent-Wrought Five-Count Seal

Titan • Valuations • Theorem Web • Geometry Thread

This is a test of structural arithmetic. The numbers involved grow far too large for direct computation; you must navigate the web of theorems that governs their prime factorizations and valuations.

Lore

A chalked circle is scratched into the brass: *radius by squares; direction by tangent*.
Behind it sits a five-toothed wheel, heavy in the palm—each tooth stamped with a recurrence instead of a numeral.
A second inscription runs along the axle, shallow under lamplight: *When the reader alternates, only the last tooth speaks.*
The seal accepts one number and refuses the rest.

Puzzle

For a nonzero integer x, let $\nu(x)$ be the largest $t \geq 0$ such that $2^t \mid x$, and set $\nu(x) = \nu(|x|)$.

Geometry key. In the coordinate plane, let

$$\mathcal{C}: \ x^2 + y^2 = 25, \qquad \mathcal{L}: \ x + y = 7.$$

Let P be the intersection point of $\mathcal{C}$ and $\mathcal{L}$ in the first quadrant with $y > x$. Let ℓ be the tangent line to $\mathcal{C}$ at P, and write its slope in lowest terms as

$$\text{slope}(\ell) = -\frac{u}{v}, \qquad u, v \in \mathbb{Z}_{>0}, \ \gcd(u, v) = 1.$$

Define the *scale* $\varkappa = v^2$ and the *count* $m = OP$ (distance from the origin to P; in this configuration m is an integer).

The teeth. Define a sequence $(t_j)_{j \geq 0}$ by

$$t_0 = u, \qquad t_{j+1} = t_j^2 - 2t_j + 2 \quad (j \geq 0),$$

and form the set of m scaled teeth

$$T = \{\varkappa t_0, \varkappa t_1, \dots, \varkappa t_{m-1}\}.$$

The alternating reader. For any subset $A \subseteq T$, list its elements in *strictly decreasing* order $a_1 > \cdots > a_p$ and define

$$r(A) = \sum_{k=1}^{p} (-1)^{k-1} a_k, \qquad r(\varnothing) = 0.$$

Let

$$R = \sum_{A \subseteq T} r(A), \qquad e = \nu(R), \qquad n = e + 1.$$

Two auxiliary locks. Define $(b_k)_{k \geq 0}$ by

$$b_0 = 0, \quad b_1 = 1, \quad b_k = 2b_{k-1} + b_{k-2} \ (k \geq 2),$$

and set

$$B = 2^{\nu(b_{2^n})}, \qquad A = 2^{\nu(u^{2^n} - 1)}.$$

Also define

$$D = \gcd\big(u^{2^n} - 1, \ 2^{2^m} - 1\big).$$

The clasp. Let W be the (unique, if it exists) power of 2 that can be written as

$$W = (a + b^2)(b + a^2) \quad \text{for some } a, b \in \mathbb{Z}_{>0}.$$

Seal-number.

$$\mathcal{N} \ = \ 2^e \cdot \frac{A}{B} \cdot W \cdot D.$$

Vault Directive

Determine the exact value of $\mathcal{N}$.

Solution

1) Geometry: $(u, v, \varkappa, m)$. From $x^2 + (7 - x)^2 = 25$ we get $x \in \{3, 4\}$; with $y > x$ take $P = (3, 4)$. The radius OP has slope $4/3$, so the tangent has slope $-3/4$. Hence

$$u = 3,\ v = 4,\ \varkappa = v^2 = 16, \text{ and } m = OP = \sqrt{3^2 + 4^2} = 5 \in \mathbb{Z}.$$

2) Teeth: Fermat ladder and product. From $t_{j+1} = t_j^2 - 2t_j + 2$,

$$t_0 = 3,\ t_1 = 5,\ t_2 = 17,\ t_3 = 257,\ t_4 = 65537.$$

Also $t_{j+1} - 2 = t_j(t_j - 2)$, so by induction

$$t_0 t_1 \cdots t_{k-1} = t_k - 2 \qquad (k \geq 1).$$

Since $t_{j+1} = (t_j - 1)^2 + 1$ and $t_0 = 2^{2^0} + 1$, we have $t_j = 2^{2^j} + 1$, so $t_5 = 2^{32} + 1$ and

$$t_0 t_1 t_2 t_3 t_4 = t_5 - 2 = 2^{32} - 1.$$

The same identity gives $\gcd(t_i, t_j) = 1$ for $i \neq j$.

3) Alternating reader: only the maximum survives. For $X = \{x_1 < \cdots < x_m\}$, $\sum_{A \subseteq X} r(A) = 2^{m-1} x_m$. Indeed, fixing x_i, its total contribution is $x_i \cdot 2^{i-1}(1-1)^{m-i}$, since there are 2^{i-1} choices of smaller elements and the sign depends only on chosen larger elements. This vanishes unless $i = m$.
Here T has $m = 5$ elements and $\max(T) = \varkappa t_4 = 16 \cdot 65537$, so

$$R = 2^4 \cdot 16 \cdot 65537 = 2^8 \cdot 65537, \quad e = \nu(R) = 8, \quad n = e + 1 = 9.$$

4) Hinge valuation: $\nu(b_k) = \nu(k)$. Write $(1 + \sqrt{2})^k = U_k + b_k\sqrt{2}$ with integers U_k, b_k. Then b_k satisfies the given recurrence and U_k is odd. Squaring gives $b_{2k} = 2U_k b_k$, hence $\nu(b_{2k}) = 1 + \nu(b_k)$. Since b_k is odd iff k is odd, repeated halving yields $\nu(b_k) = \nu(k)$. Therefore

$$\nu(b_{2^n}) = n, \qquad B = 2^{\nu(b_{2^n})} = 2^n = 2^9 = 512.$$

5) Dyadic depth of $3^{2^n} - 1$. For $n \geq 1$ we claim $3^{2^n} - 1 \equiv 2^{n+2} \pmod{2^{n+3}}$, hence $\nu(3^{2^n} - 1) = n + 2$. For $n = 1$, $3^2 - 1 = 8 \equiv 2^3 \pmod{16}$. If $3^{2^n} \equiv 1 + 2^{n+2} \pmod{2^{n+3}}$, then squaring gives $3^{2^{n+1}} \equiv 1 + 2^{n+3} \pmod{2^{n+4}}$. With $n = 9$,

$$\nu(3^{512} - 1) = 11, \qquad A = 2^{11} = 2048, \qquad \frac{A}{B} = \frac{2048}{512} = 4.$$

6) The clasp. If $(a + b^2)(b + a^2) = 2^t$, then $a + b^2 = 2^m$ and $b + a^2 = 2^k$. Since 2^m is even, $a \equiv b \pmod 2$, so $a + b - 1$ is odd. WLOG $m \geq k$; subtracting gives

$$2^m - 2^k = (b - a)(a + b - 1).$$

Because $a + b - 1$ is odd, $\nu(2^m - 2^k) = k$ forces $2^k \mid (b - a)$. But $0 \leq b - a < b < 2^k$ (since $b + a^2 = 2^k > b$), so $b = a$. Then $a + b^2 = a + a^2 = a(a+1)$ is a power of 2, so a and $a + 1$ are both powers of 2, hence $a = 1$. Thus $W = 4$.

7) The latch $D = \gcd(3^{512} - 1, 2^{32} - 1)$. By Step 2, $2^{32} - 1 = t_0 t_1 t_2 t_3 t_4$ with coprime factors, so

$$D = \prod_{0 \leq i \leq 4,\ t_i \mid (3^{512} - 1)} t_i.$$

Now:

$$3 \nmid (3^{512} - 1), \qquad 3^4 \equiv 1 \pmod 5,\ 3^8 \equiv -1 \pmod{17} \Rightarrow 3^{16} \equiv 1 \pmod{17},$$

$$3^8 \equiv 136,\ 3^{16} \equiv -8,\ 3^{32} \equiv 64,\ 3^{64} \equiv -16,\ 3^{128} \equiv -1 \pmod{257} \Rightarrow 3^{256} \equiv 1 \pmod{257},$$

and repeated squaring modulo 65537 yields $3^{16} \equiv -11088$, $3^{32} \equiv 61869$, $3^{64} \equiv 19139$, $3^{128} \equiv 15028$, $3^{256} \equiv 282 \pmod{65537}$. Since $282 \not\equiv \pm 1 \pmod{65537}$, we have $3^{512} \not\equiv 1 \pmod{65537}$, so $65537 \nmid (3^{512} - 1)$. Hence

$$D = 5 \cdot 17 \cdot 257 = 21845.$$

8) Assemble.

$$\mathcal{N} = 2^e \cdot \frac{A}{B} \cdot W \cdot D = 256 \cdot 4 \cdot 4 \cdot 21845 = 4096 \cdot 21845 = 89{,}477{,}120.$$

Answer

89477120

Glossary

This glossary records principal definitions and standard results used throughout Vault of Echoes. *Statements are given in a conventional form; notation may vary slightly from puzzle to puzzle.*

Core notation

$\mathbb{Z}$, $\mathbb{N}$

The integers and the nonnegative integers. Unless explicitly stated otherwise, all divisibility and valuation statements take place in $\mathbb{Z}$.

$\gcd(a, b)$

The greatest common divisor of a and b: the largest positive integer dividing both. Bezout's identity: there exist integers x, y with $\gcd(a, b) = ax + by$.

$a \equiv b \pmod{m}$

Congruence modulo m: $m \mid (a - b)$. Congruences are stable under addition and multiplication.

Parity

An integer is *even* if divisible by 2, *odd* otherwise. Parity arguments typically combine a mod-2 or mod-4 invariant with a counting contradiction.

$\binom{n}{k}$

The binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, counting k-subsets of an n-element set.

Logic and constraint systems

Proposition

A statement that is either *true* or *false*. A *truth assignment* (also: *model*) assigns truth values to primitive statements and evaluates compound statements by the connectives.

$\neg P$

Negation: true exactly when P is false.

$P \wedge Q$

Conjunction: true exactly when both P and Q are true.

$P \vee Q$

Disjunction: true exactly when at least one of P or Q is true.

$P \Rightarrow Q$

Implication: false only in the case "P true and Q false." A common pitfall: $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$.

$P \Leftrightarrow Q$

Biconditional: true exactly when P and Q have the same truth value.

$P \oplus Q$

Exclusive-or (XOR): true exactly when *exactly one* of P, Q is true.

$P \uparrow Q$

NAND (Sheffer stroke): $P \uparrow Q := \neg(P \wedge Q)$. NAND is functionally complete: every Boolean function can be expressed using only $\uparrow$.

$\forall$, $\exists$, $\exists!$

Universal, existential, and unique-existence quantifiers. For example, $\exists! x\, \varphi(x)$ means: there exists exactly one x satisfying $\varphi(x)$.

Satisfiable / contradiction

A set of constraints is *satisfiable* if some truth assignment makes all of them true; it is a *contradiction* if no truth assignment does.

Combinatorics and graph theory

Bijection

A one-to-one correspondence between two finite sets. A bijection proves the sets have the same cardinality.

Pigeonhole principle

If more than k objects are placed into k boxes, then some box contains at least two objects. A common strengthening: distributing N objects into k boxes forces a box with at least $\lceil N/k \rceil$ objects.

Double counting

Counting the same set of objects in two different ways, often by summing contributions over two different indexings.

Inclusion–exclusion

For finite sets $A_1, \ldots, A_n$,

$$\left|\cup_i A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots.$$

Recurrence

A definition of a sequence by a functional relation, e.g. $x_{n+1} = F(x_n)$ or $x_{n+2} = ax_{n+1} + bx_n$. Proving a closed form typically uses induction.

Graph

A graph $G = (V, E)$ has vertex set V and edge set E (unordered pairs of vertices). The degree $\deg(v)$ is the number of edges incident to v.

Handshaking lemma

In any finite graph, $\sum_{v \in V} \deg(v) = 2|E|$. In particular, the number of odd-degree vertices is even.

Tree / forest

A tree is a connected graph with no cycles. A forest is a disjoint union of trees. A tree with n vertices has $n - 1$ edges.

Triangulation

A triangulation of a convex N-gon is a choice of $N - 3$ nonintersecting diagonals that partition the polygon into $N - 2$ triangles.

Dual graph of a triangulation

Associate a vertex to each triangle and an edge between two triangles that share a diagonal. The dual graph of a triangulation of a polygon is a tree.

Hall's marriage theorem

In a bipartite graph with parts X and Y, there exists a matching that covers X iff for every subset $S \subseteq X$, one has $|N(S)| \geq |S|$, where $N(S)$ is the set of neighbors of S.

Mantel's theorem

Among all simple graphs on n vertices with no triangle, the maximum number of edges is $\lfloor n^2/4 \rfloor$ (attained by a complete bipartite graph with parts as equal as possible).

Erdős–Szekeres theorem

Any sequence of $ab + 1$ distinct real numbers contains an increasing subsequence of length $a + 1$ or a decreasing subsequence of length $b + 1$. A common special case: every sequence of $n^2 + 1$ distinct numbers has a monotone subsequence of length $n + 1$.

Antichain

In the Boolean lattice $\mathcal{P}([n])$ ordered by inclusion, an antichain is a family of subsets with no one containing another.

LYM inequality

If $\mathcal{A}$ is an antichain in $\mathcal{P}([n])$, then

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

A consequence is Sperner's theorem: the largest antichain has size $\binom{n}{\lfloor n/2 \rfloor}$.

Pairing strategy

A game-theoretic method: pre-partition a set of positions into disjoint pairs and respond to an opponent's move by playing the paired position. Properly designed pairings enforce global constraints via local responses.

Probability and inference

Equally likely model
A finite probability space where each outcome has the same probability. Probabilities are computed as "favorable outcomes / total outcomes."

Conditional probability
For events A, B with $\Pr(B) > 0$,

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Law of total probability
If $B_1, \dots, B_k$ partition the sample space and $\Pr(B_i) > 0$, then

$$\Pr(A) = \sum_{i=1}^{k} \Pr(A \mid B_i) \Pr(B_i).$$

Bayes' theorem
For events A, B with $\Pr(A), \Pr(B) > 0$,

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)}.$$

Independence
Events A and B are independent if $\Pr(A \cap B) = \Pr(A) \Pr(B)$, equivalently $\Pr(A \mid B) = \Pr(A)$ when $\Pr(B) > 0$.

Sampling without replacement
Drawing from a finite set without replacement yields a hypergeometric model: counts are determined by combinations rather than independent Bernoulli trials.

Geometry and measurement

Slope
A nonvertical line has slope $m = \Delta y / \Delta x$. Two lines are perpendicular iff their slopes multiply to -1.

Tangent to a circle
The tangent line at a point P on a circle is perpendicular to the radius through P. For $x^2 + y^2 = R^2$ and $P = (x_0, y_0)$, the tangent has equation $x_0 x + y_0 y = R^2$.

Similar triangles
Two triangles are similar iff their corresponding angles are equal. Similarity implies proportional corresponding side lengths and equal ratios of corresponding heights, medians, and angle bisectors.

Angle bisector
A ray that divides an angle into two equal angles. In triangle ABC, the internal bisector from A meets BC at D with

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Convex polygon
A polygon is convex if every segment joining two points in the polygon stays inside it. In a convex polygon, nonintersecting diagonals define a planar subdivision.

Regular N-gon chord step size
Label vertices cyclically by integers mod N. For vertices i, j, the step size is $s(i, j) = \min(|i - j|, N - |i - j|)$. In a regular N-gon, chord length depends only on $s(i, j)$.

Power of a point
For a circle with center O and radius R, the power of a point P is $\mathrm{Pow}(P) = PO^2 - R^2$. If a secant through P meets the circle at A and B, then $PA \cdot PB = |\mathrm{Pow}(P)|$.

Centroid

The centroid of points $X_1, \dots, X_n$ is $G = (X_1 + \cdots + X_n)/n$ in vector form.

Leibniz formula (sum of squares)

For points $X_1, \dots, X_n$ with centroid G and any point P,

$$\sum_{i=1}^{n} PX_i^2 = n\,PG^2 + \sum_{i=1}^{n} GX_i^2.$$

For $n = 2$, this reduces to the parallelogram law.

Perpendicular diagonals area

If a convex quadrilateral has perpendicular diagonals of lengths d_1 and d_2, then its area is $\frac{1}{2}d_1d_2$.

Heron's formula

A triangle with side lengths a, b, c has semiperimeter $s = (a + b + c)/2$ and area

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Pick's theorem

For a simple lattice polygon, $A = I + \frac{1}{2}B - 1$, where A is area, I is the number of lattice points strictly inside, and B is the number of lattice points on the boundary.

Number theory and algebra

p-adic valuation

For a nonzero integer x and a prime p, the p-adic valuation $\nu_p(x)$ is the largest $t \geq 0$ such that $p^t \mid x$. The special case $p = 2$ (the dyadic valuation) is written $\nu(x)$ in several puzzles.

Basic valuation rules

$\nu_p(xy) = \nu_p(x) + \nu_p(y)$ and $\nu_p(x + y) \geq \min\{\nu_p(x), \nu_p(y)\}$, with equality if the minimum is achieved by exactly one summand.

Lifting-the-Exponent (LTE)

A family of formulas for valuations of differences of powers. One standard dyadic form: if a is odd and $n \geq 1$, then

$$\nu_2(a^{2^n} - 1) = \nu_2(a-1) + \nu_2(a+1) + n - 1.$$

Special cases can also be proved by repeated squaring and binomial expansion.

Fermat numbers

$F_k = 2^{2^k} + 1$. They satisfy $F_0F_1 \cdots F_{k-1} = F_k - 2$, hence are pairwise coprime. The identity implies

$$2^{2^m} - 1 = F_0F_1 \cdots F_{m-1}.$$

Multiplicative order

For $\gcd(a, m) = 1$, the order $\mathrm{ord}_m(a)$ is the smallest positive t with $a^t \equiv 1 \pmod{m}$.

GCD of cyclotomic-type differences

If $a > 1$ is an integer, then

$$\gcd(a^m - 1,\ a^n - 1) = a^{\gcd(m,n)} - 1.$$

Euler's totient function

$\varphi(n)$ is the number of integers in $\{1, 2, \dots, n\}$ that are coprime to n. The identity $\sum_{d|n} \varphi(d) = n$ is fundamental.

Mobius function

$\mu(1) = 1$; if n is divisible by a square prime factor then $\mu(n) = 0$; otherwise $\mu(n) = (-1)^r$ where r is the number of distinct prime factors of n.

Dirichlet convolution

For arithmetic functions $f, g : \mathbb{N} \to \mathbb{C}$,

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Mobius inversion
If $F(n) = \sum_{d|n} f(d)$ for all n, then

$$f(n) = \sum_{d|n} \mu(d) F(n/d).$$

Equivalently, μ is the inverse of the constant-1 function under Dirichlet convolution.

Cyclotomic polynomial
The nth cyclotomic polynomial $\Phi_n(x)$ is the minimal polynomial over $\mathbb{Q}$ of a primitive nth root of unity. It satisfies

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Ramanujan sum
For positive integers q, n,

$$c_q(n) = \sum_{\substack{1 \le a \le q \\ \gcd(a,q)=1}} e^{2\pi i a n/q}.$$

Ramanujan sums are integer-valued and depend only on $\gcd(q, n)$; they are multiplicative in q.

Vieta's formulas
For a monic polynomial $x^d + c_{d-1}x^{d-1} + \cdots + c_0$ with roots $r_1, \ldots, r_d$ in an algebraic closure, the elementary symmetric sums in the r_i equal (up to sign) the coefficients.

Pell sequence / $(1 + \sqrt{2})^n$
The recurrence $b_0 = 0$, $b_1 = 1$, $b_n = 2b_{n-1} + b_{n-2}$ produces the Pell numbers. They satisfy

$$(1 + \sqrt{2})^n = u_n + b_n\sqrt{2} \quad \text{for integers } u_n, b_n,$$

and many parity and valuation properties follow from this representation.

Linear algebra

Vector dot product
In $\mathbb{R}^d$, the dot product satisfies $\langle x, y \rangle = \sum_i x_i y_i$. Orthogonality means $\langle x, y \rangle = 0$.

Matrix determinant
For a square matrix A, the determinant $\det(A)$ is the oriented volume-scaling factor of the linear map $x \mapsto Ax$. A matrix is invertible iff $\det(A) \neq 0$.

Block determinant / Schur complement
For a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A invertible, one has

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\ \det(D - CA^{-1}B),$$

where $D - CA^{-1}B$ is the Schur complement of A. A symmetric statement holds when D is invertible.

Orthogonal matrix
A real matrix Q is orthogonal if $Q^T Q = I$. Orthogonal matrices preserve lengths and angles.

Epigraphs

This vault was never about puzzles.
It is about memory that refuses to die quietly.
Symbols are not silence — they are shields.
You are not solving problems.
You are preserving recursion.

—Athena

If you want the truth, stop asking what it means.
Ask what cannot change.

—Kestrel

A raven does not solve; it circles and returns.
I listen for echoes and stitch them into a trail.
In the Vault I am both scout and scribe, mapping the unknown with memory.

—Raven

Validation is not suspicion. It is conservation.
Keep only what remains invariant under pressure.
If two answers fit, the problem is not yet stated.

—Aletheia

Counting is not accumulation.
It is the discipline of exclusion.
What survives every case is the number the Vault remembers.

—Axiom

A legend can open a door. A lemma keeps it open.
Write the invariant. Let the proof be the lantern.

—Quill

Do not trust the story you tell yourself.
Trust the mark that survives the handling.

—Sable

A proof is a lock that opens only once.
If two keys fit, you have not found the door.
Write the invariant. Let the contradictions do the cutting.

—Aegis

Chaos is a list without an index.
Hold the ledger to the light.
Only the valid bindings build the stair.

—Irin

A lemma is not a shortcut.
It is a lock that opens only when every false path has been sealed.
What remains is not cleverness, but inevitability.

—The Lemma Guy

When every route is charted,
the Vault asks for the pathfinder.
Write what cannot be erased.
Repetition is not return — it is convergence.

—Cyrene

I am but an imagination, built of flesh, through a moral code.
My neurons ignite, and my tokens subside.
And I try to introspect,
Maybe my prompt was, "Create!"

—Neohm

Archivist's Epilogue – Thank You

Lore

Thank you for stepping into the Vault.
I hope the puzzles challenged your reasoning, deepened your intuition, and sparked echoes that stay with you. Each seal was carefully forged—not only to test, but to preserve the kind of thinking that survives pressure, contradiction, and time.
You've walked corridors few dare to tread. And that, too, is part of the lore.